The History of the Law of Quadratic Reciprocity

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Chapter 1

Introduction

Pure mathematics is, in its way, the poetry of logical ideas

- Albert Einstein [1, p.viii]

After the Dark Ages, during the 17th - 19th Century, mathematics radically developed. It was a time when a large number of the most famous mathematicians of all time lived and worked. During this period of revival one particular area of mathematics saw great development, number theory. Many mathematicians played a role in this development especially, Fermat, Euler, Legendre and Gauss. Gauss, known as the Princeps Mathematicorum [2, p.1188] (Prince of Mathematics), is particularly important in this list. In 1801 he published his ground breaking work 'Disquisitiones Arithmeticae', it was in this work that he introduced the modern notation of congruence [3, p.41] (for example $-7 \equiv 15 \bmod 11$) [4, p.1]). This is important because it allowed Gauss to consider, and prove, a number of open problems in number theory. Sections I to III of Gauss's Disquisitiones contains a review of basically all number theory up to when the book was published. Gauss was the first person to bring all this work together and organise it in a systematic way and it includes a proof of the Fundamental Theorem of Arithmetic, which is crucial in much of his work. Among this work was, in Section IV, the statement and complete proof of the law of quadratic reciprocity which 'is one of the gems of eighteenth and nineteenth century mathematics' [5, p.265]. However, Gauss was not first to state the law of quadratic reciprocity, but the first to provide a rigorous proof [6, p.128], and in his lifetime he published eight distinct proofs [7]. Fermat, Euler, Lagrange and Legendre all worked in number theory and considered the law of quadratic reciprocity in their own way, and each provided work which many scholars found useful. Gauss acknowledges this in the Author's Preface to 'Disquisitiones Arithmeticae':

> *Far more is owed to modern authors, of whom those few men of immortal glory P. de Fermat, L. Euler, L. Lagrange, A. M. Legendre (and a few others) opened the entrance to the shrine of this divine science and revealed the abundant wealth within it.* [4, p.xviii]

But the history of the law of quadratic reciprocity does not stop with the proof given by Gauss in 'Disquisitiones Arithmeticae', a number of mathematicians have provided proofs of the law, in fact some 233 proofs have been published [7]. 224 of these proofs appear in Franz Lemmermeyer's 'Reciprocity Laws: From Euler to Eisenstein' and an exercise from Kenneth Ireland and Michael Rosen's book 'A Classical Introduction to Modern Number Theory' sums this up:

(**26**) Count the number of proofs to the law of quadratic reciprocity given thus far in this book and devise another one. [8, p.202]

Once there was a solution for quadratic reciprocity, people, including Gauss, started looking for solutions to cubic reciprocity, quartic reciprocity and higher powers. But the law of quadratic reciprocity is not just an academic questions in number theory, it has far reaching consequences, for example in cryptography. You can construct cryptographic schemes by finding the square root modulo large composite numbers and in integer factorization where you can construct factoring algorithms which use the law of quadratic reciprocity [4, p.396]. Also in acoustics, Quadratic-Residue Diffusers are widely used, they can diffuse sound in one or two directions [9].

Chapter 2

An Idea Is Born

Problems worthy of attack prove their worth by hitting back

- Piet Hein [10, 401]

Pierre de Fermat, the Father of Modern Number Theory [11, p.xiii], was born in 1601 or 1607/08 [12] in Beaumont-de-Lomagne, France [13]. He attended the University of Toulouse studying Civil Law and graduated in 1631 [13, p.15]. Fermat was very talented with languages and was fluent in many, including Greek. It was this mastery of the language which was the reason he was often asked to help with the revision of Greek texts [13, p.3]. Fermat was an amateur mathematician, but none the less he worked on a broad range of mathematics, including analytic geometry. His work on the subject was being read from 1637 [14, p.704] which was at the same time as Descartes's, 'La géométrie', however it was not officially published until 1679, after his death [15, p.548]. Fermat also developed a method for calculating maxima, minima and tangents to various curves [16, p.358-362] [14, p.741], this, along with the calculation of the integral of a general power function, was helpful to both Newton and Leibniz in their development of calculus [14, p.741] [17, p.98]. But one area which Fermat had a massive impact was number theory. He studied Pell's Equation [16, p.364], numbers which are now known as Fermat Numbers and Fermat Primes [16, p.361], he developed his 'Little Theorem' [16, p.361] and the world famous 'Last Theorem' [18, p.1]. Fermat stated a lot of theorems and conjectures but did not provide proofs for all of them, instead this was left to his successors, Euler, in particular, proved many of these assertions. Before we can explore what Fermat contributed to the history of the law of quadratic reciprocity we must introduce the concept of a *quadratic residue*. An integer r is a quadratic residue of m if a square, x^2, exists such that:

$$x^2 \equiv r \mod m \, [19, p.201]$$

Fermat considered a number of theorems regarding when you can express a prime in quadratic form. He proved [20, p.2], although Gauss says he did not and only conjectured [4, p.148], that for a prime p:

$$p = x^2 + y^2 \quad \text{if and only if} \quad p = 2 \text{ or } p \equiv 1 \mod 4$$

After this, Fermat also considered similar theorems for primes of the form $x^2 + ny^2$ for $n = \pm 2$, ± 3, -5 [20]. Although Fermat never stated the law of quadratic reciprocity [21, p.62], the cases for -1, ± 2 and ± 3 can be found from his theorems. It was his early studies which Euler then continued [20, p.2-5], that led mathematicians in the direction of the law of quadratic reciprocity.

Chapter 3

Our Master In Everything

He calculated without any apparent effort, just as men breathe, as eagles sustain themselves in the air

- François Arago [22, 440]

Leonhard Paul Euler was born in Basel, Switzerland, on April 15th 1707 [14, p.747] to Paul Euler and Marguerite Brucker [23, p.143]. Paul Euler was a friend of Johann Bernoulli [24, p.6], who at the time was thought to be the leading mathematician in the world [25, p.252] and who would turn out to have a great influence on the life of Leonhard Euler [24]. At the age of 13, Euler started at the University of Basel, studying Philosophy, and after three years completing his degree [26, p.47] with a dissertation comparing the philosophies of Descartes and Newton [27, p.161]. It was during this time that Euler received lessons from Johann Bernoulli, who noticed Euler's incredible talent for mathematics [28, p.2]. Euler left Switzerland in 1727 and took a position at the 'Imperial Russian Academy of Sciences' in St. Petersburg [14, p.749], by 1733 he had succeeded Daniel Bernoulli, the son of Johann, to the position as head of the mathematics department [29]. Euler then left Russia in 1741 to start work at the 'Berlin Academy', where he spent the next twenty-five years of his life and published over 380 articles [14, p.749]. For long periods of Euler's life he had trouble with his sight [30, p.334] [31, p.374] [14, p.748], in 1738, three years after suffering a nearly fatal fever, he lost vision in his right eye [30] [31], and then in 1766 a cataract was discovered in his left eye making him practically blind for the rest of his life [14, p.749]. Perhaps testimony to Euler's ability is that this loss of sight did not stop his richness of work [14, p.749] and on average he produced a mathematical paper a week in 1775 [32]. Euler worked and revolutionized almost all areas of mathematics: geometry, calculus [14, p.749] (in particular his paper entitled 'Institutiones calculi differentials' published in 1755 [33]), trigonometry and algebra (with one of his most famous works 'Introductio in Analysin Infinitorum' published 1748 [14, p.749]), lunar theory [14, p.749] and other areas of physics. Euler also had a great interest in number theory and he seemed to have begun reading Fermat's work after he started speaking to Christian Goldbach [20, p.3], and a lot of his early work was spent proving theorems and conjectures that Fermat had stated, but not published proofs for [14, p.747-748]. Euler's first proof regarding the law of quadratic reciprocity [20, p.4] is now known as Euler's Criterion and states that:

Euler's Criterion. For integers a and odd primes p such that $p \not| a$ we have:

$$a^{\frac{p-1}{2}} \begin{cases} +1 \mod p, \text{ if a is a quadratic residue mod } p, \\ -1 \mod p, \text{ if a is a quadratic non-residue mod } p\,[20, p.4] \end{cases}$$

This is the first of a number of results which Euler, and other mathematicians, have stated and proved which led to the first full statement of the law of quadratic reciprocity, or an equivalent statement, and later its proof. It was in 1744 that the first statement equivalent to the quadratic reciprocity law was stated by Euler [34, p.4], Euler worked on the theorems that Fermat had discovered and he soon came up with, translated into English, and in a more modern notation:

$$\text{The quadratic residue character of } p \text{ modulo primes of the form:}$$
$$4pn \pm s \text{ with } 0 < s < 4p \text{ and } (s, 2p) = 1 \text{ does not depend on s.}[20, 4]$$

Joseph Lagrange, born in 1736 [35, p.153], succeeded Euler as head of mathematics at the 'Imperial Russian Academy of Sciences' [36, p.145], knew of Euler's work and between 1773 and 1775 [20] worked on this result. Lagrange's work inspired Euler to continue working in this particular area and he came about the complete law of quadratic reciprocity [20, 5], however this was not published until 1783 [37], after his death [14, p.747], and Euler never found a proof for it. Though, if it was not for the remarkable work that Euler did in this area the world may have had to wait much longer for the law of quadratic reciprocity, because although it was Gauss that first proved this law, it was Euler that 'discovered' it. In modern notation, translated from the original Latin, Euler stated:

1. If $p \equiv 1 \mod 4$ is prime and $p \equiv x^2 \mod s$ for some prime s, then $\pm s \equiv y^2 \mod p$.

2. If $p \equiv 3 \mod 4$ is prime and $-p \equiv x^2 \mod s$ for some prime s, then $s \equiv y^2 \mod p$ and $-s \not\equiv y^2 \mod p$.

3. If $p \equiv 3 \mod 4$ is prime and $-p \not\equiv x^2 \mod s$ for some prime s, then $s \equiv y^2 \mod p$ and $-s \not\equiv y^2 \mod p$

4. If $p \equiv 1 \mod 4$ is prime and $p \not\equiv x^2 \mod s$ for some prime s, then $\pm s \not\equiv y^2 \mod p$. [20, p.4-5]

Chapter 4

The Man With No Face

Our colleague has often expressed the desire that, in speaking of him, it would only be the matter of his works, which are, in fact, his entire life

- Siméon-Denis Poisson [38, 114]

The last person to consider before Gauss gave the world notation that is still used today regarding the law of quadratic reciprocity and stated a formula which is often explicitly calculated to prove quadratic reciprocity. Adrien-Marie Legendre was born in Paris, 1752 [14, p.754], although some state this as Toulouse [39, p.421]. The Legendre family was a wealthy one, and Adrien-Marie received a good education at the 'Collége Mazarin' [40, p.60]. After completing his degree he taught at 'École Militarie' for five years from 1775 [40]. Legendre worked in many areas of mathematics and in 1782 he won a prize from the 'Berlin Academy' for his work on ballistics [40], this was followed by being appointed as the successor to Lagrange at the 'Académie des Sciences' in 1783. He lost his family fortune during the French Revolution in 1793 [40] but he managed to organise his financial affairs and in 1795 became a member of the mathematical section of the 'Académie des Sciences' [41, p.136]. Like Fermat and Euler, Legendre worked in many areas of mathematics, his 'Éléments de géométrie' was a leading text on elementary geometry for over 100 years from its publication in 1794 [14, p.755]. He also did a lot of work on elliptical functions [14, p.754], however the majority of his work was in number theory. He considered the distribution of primes [42] and the applications of analysis to number theory which led him to the statement of the Prime Number Theorem in 1796 [43, p.15]. In 1830 he added his name to the list of people who considered Fermat's Last Theorem by solving it for the case $n = 5$ [44, p.85]. Legendre also did a lot of work on the law of quadratic reciprocity, and a lot of what he did, in notation and statement, we still use today.

He begins one of his most famous papers, 'Recherches d'analyse indéterminée, Histoire de l'Académie' [45] with a proof of one of Euler's results [46]. In Article IV, letting a and A represent positive primes congruent to 1 modulo 4 and b and B represent positive primes congruent to 3 modulo 4, states eight theorems on pages 516 and 517 [20, p.6] of 'Recherches d'analyse' which are equivalent to quadratic reciprocity. The following are the theorems translated from the original French, and in modern notation (Legendre used $=$ instead of $\equiv$):

Theorem 1 *If* $b^{\frac{a-1}{2}} \equiv +1$, *it follows* $a^{\frac{b-1}{2}} \equiv +1$

Theorem 2 *If* $a^{\frac{b-1}{2}} \equiv -1$, *it follows* $b^{\frac{a-1}{2}} \equiv -1$

Theorem 3 *If* $a^{\frac{A-1}{2}} \equiv +1$, *it follows* $A^{\frac{a-1}{2}} \equiv +1$

Theorem 4 *If* $a^{\frac{A-1}{2}} \equiv -1$, *it follows* $A^{\frac{a-1}{2}} \equiv -1$

Theorem 5 *If* $a^{\frac{b-1}{2}} \equiv +1$, *it follows* $b^{\frac{a-1}{2}} \equiv +1$

Theorem 6 *If* $b^{\frac{a-1}{2}} \equiv -1$, *it follows* $a^{\frac{b-1}{2}} \equiv -1$

Theorem 7 *If* $b^{\frac{B-1}{2}} \equiv +1$, *it follows* $B^{\frac{b-1}{2}} \equiv -1$

Theorem 8 *If* $b^{\frac{B-1}{2}} \equiv -1$, *it follows* $B^{\frac{b-1}{2}} \equiv +1$

Here, we must note that in fact Theorems 1 and 2 are essentially the same, as are Theorems 3 and 4, and Theorems 5 and 6 [45], so in fact there are only five different cases. From this point onwards in the paper Legendre goes about trying to prove these theorems. He mentions that he is unable to completely prove Theorems 1, 3 and 7 [46], however his other proofs are still dependent on another theorem, which he could not prove. This is now known as Dirichlet's Theorem:

Dirichlet's Theorem. Let a and b be positive integers; if $gcd(a, b) = 1$, then there exist infinitely many primes $\equiv a \mod b$ [20, p.8]

It is so named as it was first solved by Dirichlet in 1837 [47], so although Legendre could not provide a rigorous proof of these theorems, he had successfully stated the law of quadratic reciprocity. His second major paper 'Essai sur la Théorie des Nombres' [46] from 1798 [20, p.6] introduced some crucial notation, which is now called the Legendre Symbol [45], which although did not introduce anything new, made performing certain calculations easier, the Legendre Symbol is defined as:

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if}\, a \,\text{is a quadratic residue modulo}\, p \\ 0 & \text{if}\, p \mid a \\ -1 & \text{otherwise}\, [48, p.117] \end{cases}$$

Legendre also gives us something else, the word 'reciprocity' [46] (in a mathematical way) which, clearly, is still used today. In the paragraph (164 [45]) in which Legendre first uses the term Reciprocity he states, translated from the original French:

Whatever the primes m and n, if not both of the form $4x - 1$, we will always have that:

$$\left(\frac{n}{m}\right) = \left(\frac{m}{n}\right)$$

And if both are of the form $4x - 1$, we have that:

$$\left(\frac{n}{m}\right) = -\left(\frac{n}{m}\right)$$

These two general cases are included in the formula:

$$\left(\frac{n}{m}\right) = (-1)^{((\frac{m-1}{2})\cdot(\frac{n-1}{2}))}\left(\frac{m}{n}\right) [46]$$

This is very important result since a lot of the proofs of quadratic reciprocity now directly calculate this formula (or an equivalent version) [7]. Again, Legendre goes about trying to prove this result, and in doing so proving quadratic reciprocity. However he once again does not arrive with a general, rigorous proof, which does not rely on Dirichlet's Theorem, and his proof is basically the same as his earlier attempts [45]. However the notation he introduced has not only led to a number of ways of proving quadratic reciprocity it has had a lasting impact on the history of higher reciprocity laws. One of these ways is through the development of the Jacobi and Hilbert symbols which have come about through the study of the Legendre symbol [45].

Chapter 5

Princeps Mathematicorum

Mathematics is the queen of the sciences and arithmetic the queen of mathematics

- Carl Friedrich Gauss [19, p.vi]

Johann Carl Friedrich Gauss, known as Carl Gauss, was born April 30th 1777 in Braunschweig [14, p.755], which is now a part of Lower Saxony in Germany. There are many stories of the brilliance that Gauss showed at an early age, including mentally correcting a mistake of his father's in his finances [25, p.334]. There is also the now famous story that Gauss was asked to find the sum of the first 100 counting numbers, which he quickly determined to be 5050 [49, p.97]. However we should perhaps not take these stories as completely accurate, as Gauss was known to exaggerate them [50, p.287]. Gauss was supported by the Duke of Braunschweig [51] who gave him a fellowship, he later went to the University of Göttingen from 1795 to 1798. It was during this time at university that Gauss had a remarkable year, 1796, and worked a lot in what he called 'Higher Arithmetic' [19, p.200]. On March 30th 1796 he worked out how, only using a compass and straightedge, to construct a regular heptadecagon [52, p.383]. He developed modular arithmetic in the same year, and on April 8th became the first person to prove the law of quadratic reciprocity [20, p.21]. By the end of May he was one of several mathematicians that conjectured the prime number theorem [53, p.382] and July saw him find the result that every positive integer is equal to the sum of at most three triangular numbers [54]. This remarkably successful year concluded with a result on how many solutions to polynomials with finite field coefficients there are [55].

Gauss did not solely work in number theory, but also contributed to other areas of mathematics and physics. At the age of 23 he tackled the problem of finding the dwarf planet Ceres, using very little information, he predicted the position of the planet in 1802 [56, p.228]. This eventually led Gauss to publish a paper in 1809 on the motion of planetoids displaced by large planets [57]. He also worked on non-Euclidean geometry, but for fear of controversy never published his work in this area [16, p.523].

Possibly Gauss's seminal work was his 'Disquisitiones Arithmeticae' (Latin for 'Investigations in Arithmetic' [58, p.711]) which was written in 1798 and published in 1801 [59, p.515]. The book is written in a way that set a standard for most later texts, it states a theorem, proves it and then states and proves corollaries [60], Gauss also gives numerical examples of theorems. The first three sections of the book contain few new results, and serves the purpose to bring the reader up to date with elementary number theory. This section brings us the the first correct proof of the fundamental theorem of arithmetic [61, p.54] which is highly important and used throughout mathematics [8]. Section IV contains the first complete proof of quadratic reciprocity [4, p.73-111], which was in a similar vein to the proof attempted by Legendre, although Gauss discovered this proof independently of Legendre's work [20, p.21]. This first proof is long and ugly [62, p.179] and, like Legendre, required the use of an extra result. However, unlike Legendre, Gauss was able to prove this:

Lemma: If $p = 1 \bmod 8$ is prime, then there exists an odd prime $q < 2\sqrt{p} + 1$ such that $\left(\frac{p}{q}\right) = -1$ [20, p.68]

The proof of which appears in 'Disquisitiones Arithmeticae'. The complete proof of quadratic reciprocity is done by induction, assuming true for all numbers less than n [63, p.251] (to see an updated version of this first proof read [64]). Gauss called the law of quadratic reciprocity 'Theorema Aureum' (the golden theorem) and 'Gem of the Higher Arithmetic' [19, p.200]. Gauss's diary indicates he found the complete proof to quadratic reciprocity on April 18th, 1796 [62] [40]. However the Disquisitiones was not published until 1801, and he managed to find another three proofs [7] of the law before then.

Section V of the Disquisitiones is about the theory of binary quadratic forms [22, p.500]. Binary quadratic forms are homogeneous polynomials of degree two in two variables, that is of the form:

$$f(X,Y) = aX^2 + bXY + cY^2 \quad a, b, c \in \mathbb{Z}\,[65, p.1]$$

This section is over half of the book, and in it Gauss developed genus theory of quadratic forms which he then uses to again prove the law of quadratic reciprocity. The crux of the argument is to show that two inequalities on the number of genera hold [20, p.21]. In article 366 of the Disquisitiones Gauss talks of a third proof [20, p.21], however this would have been in Section VIII, which before publication Gauss removed from the Disquisitiones, and this was not published until after his death [66]. These proofs were based on quadratic period equations [20, p.21], and, although Gauss came up with them between the years 1796 and 1801, they were not published until 1863 [7], eight years after his death [4, p.755]. However Gauss did not stop there, he found a further four proofs in his lifetime, which means that in total he found eight proofs of the law of quadratic reciprocity.

Once Gauss had presented the first proof, he started to consider higher powers of reciprocity. That is when the equation:

$$x^n \equiv m \mod p$$

where $m,\, n \in \mathbb{Z},\; p$ is a prime, and $m \nmid p$ [35, p.325]

has a solution. Also he studied the connections between:

$$x^m \equiv p \mod q \quad \text{and} \quad x^m \equiv q \mod p$$

where $m,\, n \in \mathbb{Z},\; p$ is a prime, $m \nmid p$ and p and q are rational primes [35, p.325]

$n = 2$ is quadratic reciprocity, and for $n = 4$ the term biquadratic reciprocity is used [35, p.325]. Gauss arrived at the statements of cubic and biquadratic reciprocity, and although he provided a proof for cubic reciprocity, he was not the first to publish one, that honour fell to Eisenstein in 1844 [67] [68]. Gauss said that cubic and biquadratic reciprocity was 'far more difficult' [8, p.108] than that of quadratic reciprocity and that to deal with these higher powers of reciprocity, a theory of algebraic numbers is needed. For biquadratic reciprocity Gauss considered the ring $\mathbb{Z}[i] = \{a + bi \,|\, a, b \in \mathbb{Z}\}$ [8, p.108] which is now known as the Gaussian Integers.

Chapter 6

In General

There is one word which may serve as a rule of practice for all one's life - reciprocity

- Confucius [69, 154]

The law of quadratic reciprocity has come to the attention of some of the greatest mathematicians who have ever lived. Many have considered it, and many have published proofs, the total of 233 proofs with two in preprint and one set to appear [7] are testimony to this. The history of the law is diverse, with many people contributing to it but none more so than the four mathematicians mentioned here, Fermat, Euler, Legendre and Gauss. Fermat's work on expressing primes in the form $x^2 + ny^2$ helped lead Euler to the first statement of the law of quadratic reciprocity. Legendre provided very convenient notation, and restated the law, and it was Gauss who published the first proof of the law. But who should get the credit for discovering quadratic reciprocity? Euler for being the first to present a full statement of the law, or Gauss for first providing a complete proof. A crossword clue from the New York Times highlights this question, it reads:

30. Discoverer of the law of quadratic reciprocity (5 Letters) [70]

Unfortunately for the reader, both Euler and Gauss have five letters in their name! The crossword gives the answer as Euler, but I would personally credit Gauss, since before there was a proof it could not be called a law as it was not proven to be true. Henry Smith, the British number theorist, calls this first proof ‘repulsive’ [71, p.59]. However, quadratic reciprocity is fundamental in many areas of mathematics, which is why there are so many proofs and it seems it was only a matter of time until someone found something more appealing about Gauss's first proof. That was the role of John Tate, a student of Emil Artin [72]. He noticed that Gauss's first proof can be used in K - Theory, in particular the construction of $K_2(\mathbb{Q})$ [20, p.21]. Tate used the Lemma which Gauss used in his proof for an auxiliary prime (as stated earlier) to prove the product formula for the Hilbert symbol is equivalent to the quadratic reciprocity law.

Not long after Gauss's work the question of proving the general reciprocity law arose. In 1850 [73, p.122] Eisenstein published his law of reciprocity which states:

Eisenstein's Reciprocity Law. Let l be an odd prime and suppose that $\alpha \in \mathbb{Z}[\zeta_l]$ is primary, then:

$$\left(\frac{\alpha}{a}\right)_l = \left(\frac{a}{\alpha}\right)_l$$

for all integers $a \in \mathbb{Z}$ prime to l [20, p.vii]

Which we now know is a special case of the law of general reciprocity. Many other mathematicians worked on this problem, including Hilbert, whose famous list of 23 unsolved problems, which he published after speaking at the International Congress of Mathematicians in 1900 [58, p.877], are still quoted today. The ninth problem states:

(**9**) *For any field of numbers the law of reciprocity is to be proved for the residues of the l-th power*, when l denotes an odd prime, and further when l is a power of 2 or a power of an odd prime. [74]

Or, more simply:

(**9**) Find the most general law of reciprocity in an arbitrary algebraic number field.

This was partially solved by Emil Artin between the years 1924 and 1930, in a series of papers where he developed his own reciprocity law which states:

Artin Reciprocity Law. Let K be an algebraic number field and let L/K be a finite Galois extension. Then the global norm residue symbol $\left(\frac{L/K}{\cdot}\right)$ induces an isomorphism

$$C_K/N_{L/K}C_K \cong Gal(L/K)^{ab}$$

where $G^{ab} = G/G^{'}$ is G made abelian. [75] [76] [77]

Dealing with reciprocity over all abelian extensions of algebraic number fields and, together with Teiji Takagi and Helmut Hasse's work, developed class field theory [78, p.1]. However this is not the complete solution to Hilbert's Ninth Problem, and, as of yet, there is no expansion of Artin's reciprocity law to polynomials with a non-abelian Galois extension [79, p.318]. The law of quadratic reciprocity, first proved over two hundred years ago has held the attention of the world's greatest mathematicians. It has led to the development of a whole new field of mathematics and to a phenomenal number of proofs, and it is still providing mathematicians today with questions that wait to be answered.

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