UNIVERSITY OF BIRMINGHAM

MSCI MATHEMATICS PROJECT

Real and Complex Reflection Groups

Author:
Sean ELVIDGE
885704

Supervisor:
Dr. Simon GOODWIN

April 2011.

Abstract

> Mathematics possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature ... capable of a stern perfection such as only the greatest art can show.
>
> Betrand Russell [37]

The aim of this project is to build up the theory of reflection groups for a graduate of mathematics reader and to compute a number of results regarding complex reflection groups.

Real reflection groups were first classified by Harold Coxeter in 1934 [16], indeed, real reflection groups are often called finite Coxeter groups. The theory of complex reflection groups followed in 1954 with the Shepherd-Todd theorem on invariant theory of finite groups. Recently, the theory of complex reflection groups has generated a great deal of research interest, and is central to many modern developments in algebra.

This project begins with some preliminary results in linear algebra and group theory, which are used throughout. Chapter 2 builds up the theory of roots and root systems for real reflection groups and Chapter 3 contains the classification of real reflection groups. The complex reflection groups are split into three infinite families and 34 exceptional cases. In this project we prove the classification of imprimitive complex reflection groups, which is found in Chapter 4. Although we do not prove the primitive case, we discuss such groups and reference where a proof can be found.

Chapter 5 then contains computational results regarding complex reflection groups using the computer algebra program GAP. The project is concluded in the final chapter by discussing modern developments in this field. The provided references should be a good basis for the reader to develop their understanding of reflection groups further and to study them in greater depth.

The completion of this project has only been possible with the help of my supervisor, Simon Goodwin. I wish to express my deepest thanks for his valuable time and support.

Contents

Chapter 1

Preliminaries

> The human mind has never invented a labour-saving machine equal to algebra.
>
> — Anon

1.1 Linear Algebra

First we introduce some basic notation and definitions from linear algebra. As we will in Chapter 2, here we restrict our attention to the real numbers, $\mathbb{R}$.

Definition 1.1.1. A *bilinear form* on a vector space V is a form that is linear in both parts, i.e. a function $\beta : V \times V \longrightarrow \mathbb{R}$ such that

1. $\beta(u + u', v + v') = \beta(u, v) + \beta(u, v') + \beta(u', v) + \beta(u', v')$,
2. $\beta(\lambda u, v) = \beta(u, \lambda v) = \lambda \cdot \beta(u, v)$.

A symmetric bilinear form satisfies

$$\beta(u, v) = \beta(v, u),$$

for all $u, v \in V$.

Definition 1.1.2. A *Euclidean vector space* is a set V over a field $\mathbb{R}$, together with two binary operations that satisfy the axioms of a vector space and has an *inner product*. That is, a symmetric bilinear form:

$$(\cdot, \cdot) : V \times V \longrightarrow \mathbb{R},$$

satisfying the conditions for a symmetric bilinear form and, for all $v \in V$,

$$(u, u) \geq 0 \text{ where equality holds if and only if } u = 0.$$

Example 1.1.3. If we let $V = \mathbb{R}^n$ and equip it with the usual dot product:

$$((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + \cdots + x_n y_n,$$

then V is a Euclidean vector space.

Definition 1.1.4. Let V be a Euclidean vector space, the *length* of a vector α is $\|\alpha\| := \sqrt{(\alpha, \alpha)}$.

Note that $\|\alpha\| = 0$ if and only if $\alpha = 0$. From here onwards we always let V represent a Euclidean vector space. Central to our study of reflections and reflection groups is that of a hyperplane.

Definition 1.1.5. A *hyperplane* in a vector space V is an $n - 1$ dimensional subspace of V.

Notationally we write H_α to represent the hyperplane orthogonal to a vector $\alpha \in V$. We are now in a position to define an orthogonal transformation, and eventually the orthogonal group $O(V)$.

Definition 1.1.6. The set of linear transformations, S, from V to V which satisfy:

$$(Sv,\, Sw) = (v,\, w) \quad \text{for all } \ v, w \in V,$$

is denoted by $O(V)$. The members of $O(V)$ are called *orthogonal* transformations.

Recall that a matrix A is orthogonal if $A^T = A^{-1}$, where A^T is the transpose of the matrix.

Definition 1.1.7. Two vectors in a Euclidean vector space are *orthonormal* if they are orthogonal and of unit length. A set of vectors is *orthonormal* if all the vectors in the set are mutually orthonormal and an *orthonormal basis* is an orthonormal set which forms a basis of the Euclidean vector space.

Example 1.1.8. The standard basis of $\mathbb{R}^3$ is an orthonormal basis. To see this consider the standard basis of $\mathbb{R}^3$, $\{e_1, e_2, e_3\}$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ \text{ and } \ e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Clearly $(e_1, e_2) = (e_1, e_3) = (e_2, e_3) = 0$ and $\|e_1\| = \|e_2\| = \|e_3\| = 1$. Thus $\{e_1, e_2, e_3\}$ is an orthonormal set. But since we already know that this is a basis of $\mathbb{R}^3$ we have that $\{e_1, e_2, e_3\}$ is an orthonormal basis.

Lemma 1.1.9. *Let $S \in O(V)$ and let A be the matrix that represents S with respect to an orthonormal basis, then* $\det A = \pm 1$.

Proof. Let I represent the identity matrix then, since the elements of $O(V)$ are orthogonal we get that $A^T A = I$ and thus,

$$\begin{aligned} 1 &= \det I, \\ &= \det(A^T A), \\ &= (\det A^T)(\det A), \\ &= (\det A)^2. \end{aligned}$$

Giving $\det A = \pm 1$. $\square$

1.2 Group Theory

We recall here the definition a group, a subgroup and the criterion a set with a binary operation is a subgroup.

Definition 1.2.1. A *group* is a set G with a binary operation, $\cdot$, such that:

1. For all $a, b, c \in G$ the associativity law holds, i.e. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
2. There exists an identity element, $e \in G$, i.e. for all $a \in G$, $a \cdot e = e \cdot a = a$.
3. For each $a \in G$, there exists an inverse element b, i.e. $a \cdot b = b \cdot a = e$.

We usually omit the $\cdot$ when talking about multiplication of group elements, that is, for $g, h \in G$ we write gh to represent the multiplication $g \cdot h$ in the group G.

Example 1.2.2. $\mathrm{GL}(V)$, the set of all automorphisms (bijective linear transformations $V \longrightarrow V$) of a vector space V, with composition of functions as the binary operation, is a group.

Definition 1.2.3. A nonempty subset H, of a group G, which is itself a group under the binary operation of G, is called a *subgroup* of G. We write $H \leq G$.

Lemma 1.2.4. *Let H be a non-empty subset of a group G. Then H is a subgroup of G if and only if $gh^{-1} \in H$ for all $g, h \in H$.*

We do not give a proof here, it is fairly trivial, and can be found in any good introduction to group theory book. Our intuition of $O(V)$, along with matrix multiplication, should be that it is a group, namely a subgroup of $\mathrm{GL}(V)$.

Proposition 1.2.5. *$O(V)$ is a subgroup of $GL(V)$.*

Proof. We can equivalently define the orthogonal group as $O(V) := \{A \in \mathrm{GL}(V) \,|\, A^T A = AA^T = I,$ where I is the identity matrix$\}$ and can prove it is a subgroup by using Theorem 1.2.4.

$$(AB)^T(AB) = (B^T A^T)(AB) = B^T A^T AB = B^T IB = B^T B = I,$$

thus $O(V) \leq \mathrm{GL}(V)$. □

Definition 1.2.6. For a group G and a set X we define a *group action* of G on X to be a function:

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g \cdot x \end{aligned}$$

Satisfying:

1. $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$,
2. $e \cdot x = x$ where e is the identity in G, for all $x \in X$.

Definition 1.2.7. Given a group G acting on a set X (i.e. we have a group action as defined above) the *orbit* of an element $x \in X$ is the set of elements which x is moved to by G, i.e. it is the set,

$$Gx = \{g \cdot x \,|\, g \in G\}.$$

We call the group action *transitive* if the orbit of an element $x \in X$ is the whole set, $Gx = X$.

Chapter 2

Real Reflection Groups

> Thou in thy lake dost see
> Thyself.
>
> J. M. Legaré [33, pg. 10]

2.1 Reflections

Before studying complex reflection groups we first consider real reflection groups and their classification. In this chapter we work towards the presentation of finite reflection groups.

Definition 2.1.1. For $0 \neq \alpha \in V = \mathbb{R}^n$ the *reflection* about α is the unique linear transformation mapping α to its negative and fixing pointwise the hyperplane H_α. That is the hyperplane orthogonal to α.

Remark. Each reflection s_α uniquely determines a reflecting hyperplane H_α, and vice versa.

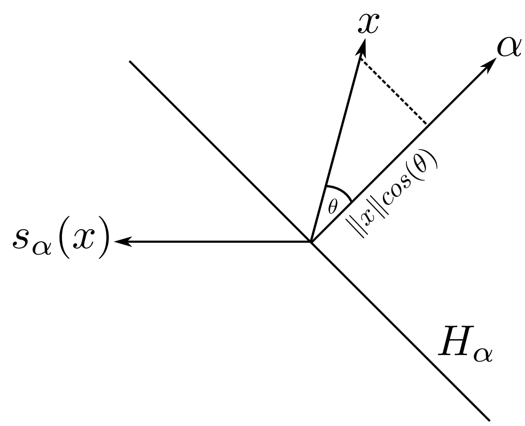


Figure 2.1: The reflection of x along α in $V = \mathbb{R}^2$.

In Figure 2.1 we would say that $s_\alpha(x)$ (which we will often just write as $s_\alpha x$) is reflected through the hyperplane H_α, or along α.

Clearly, for any reflection, $s_\alpha^2 = 1$ (reflecting the reflection will go back to where we started) and there is a simple formula for $s_\alpha x$ with $x \in V$.

Theorem 2.1.2. *Let $s_\alpha x$ be a reflection of x along α, fixing the hyperplane H_α, then:*

$$s_\alpha x = x - \frac{2(x, \alpha)\alpha}{(\alpha, \alpha)}.$$

Proof. For this proof we consider α as a unit vector,

$$\begin{aligned} \hat{\alpha} &= \frac{\alpha}{\sqrt{(\alpha, \alpha)}}, \\ &= \frac{\alpha}{\|\alpha\|}. \end{aligned}$$

Then $(x, \hat{\alpha}) = \|x\| \cos(\theta)$, (which is labelled in Figure 2.1, where θ is the angle between the vectors α and x) so:

$$\begin{aligned} s_\alpha x &= x - 2\|x\|\cos(\theta)\hat{\alpha}, \\ &= x - 2\,(x, \hat{\alpha})\,\hat{\alpha}, \\ &= x - \frac{2(x,\alpha)}{\|\alpha\|} \cdot \frac{\alpha}{\|\alpha\|}, \\ &= x - \frac{2(x,\alpha)\alpha}{(\alpha,\alpha)}. \end{aligned}$$

□

It is easy to see that $s_\alpha \in O(V)$, we can convince ourselves geometrically or using the above formula and the inner product on V to show that:

$$\begin{aligned} \left(x - \frac{2(\alpha,x)\alpha}{(\alpha,\alpha)}, y - \frac{2(\alpha,y)\alpha}{(\alpha,\alpha)}\right) &= (x,y) - \frac{2(\alpha,x)(\alpha,y)}{(\alpha,\alpha)} - \frac{2(\alpha,y)(x,\alpha)}{(\alpha,\alpha)} \\ &\quad + \frac{4(\alpha,x)(\alpha,y)(\alpha,\alpha)}{(\alpha,\alpha)^2}, \\ &= (x,y). \end{aligned}$$

So, $s_\alpha \in O(V)$. Using the fact that $s_\alpha^2 = 1$, note that s_α has order 2 in $O(V)$.

Definition 2.1.3. A finite subgroup of $O(V)$ generated by reflections is called a *real (finite) reflection group.*
We denote by W a finite reflection group, acting on the Euclidean space V.

W is used to denote finite reflection groups since the majority of finite reflection groups are actually 'Weyl groups' [30, pg. 6]. From here onwards W is used exclusively to represent some finite reflection group.

Example 2.1.4. The dihedral group, D_{2n}, is the set of orthogonal transformations which preserve a regular n-sided polygon centred at the origin. The group has order $2n$ and is of type $I_2(n)$, it consists of n rotations (rotating the shape through $\frac{2\pi}{n}$ multiple times) and n reflections. However any rotation can in fact be generated by the product of two reflections, and D_{2n} is a finite reflection group.

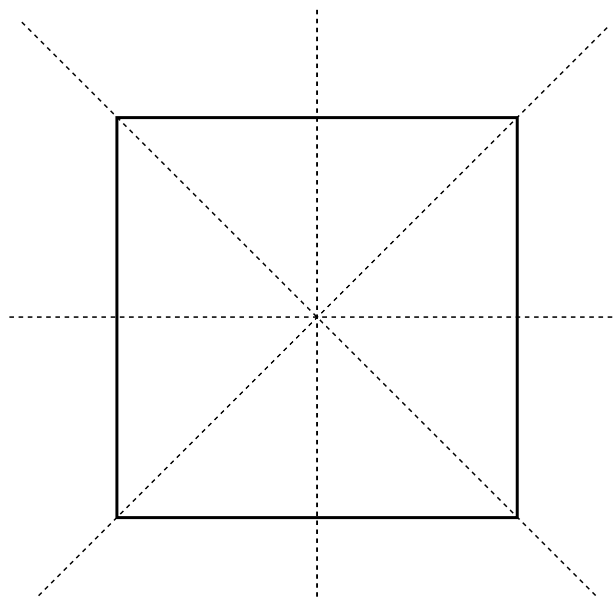


Figure 2.2: D_8, example of finite reflection group.

Example 2.1.5. The symmetric group, $\mathrm{Sym}(n)$, is the group of permutations on n letters. The group has order $n!$ and is of type A_{n-1}. We can consider the group as a subgroup of $O(\mathbb{R}^n)$ and a permutation acts on $\mathbb{R}^n$ by permuting the standard basis vectors of $\mathbb{R}$ (the set $\{e_1, \ldots, e_n\}$). The transposition (i, j) sends $e_i - e_j$ to its negative and fixes the orthogonal complement. Since we know that the symmetric group is generated by transpositions (see [2, pg. 16]) we have that the symmetric group is a reflection group.

Theorem 2.1.6. *If $T \in O(V)$ and $0 \neq \alpha \in V$, then*

$$Ts_\alpha T^{-1} = s_{T\alpha}.$$

Proof. Consider $x \in V$. We note that since $(x, \alpha) = (Tx, T\alpha)$, x lies in H_α if and only if Tx lies in $H_{T\alpha}$. Then, whenever x lies in H_α, we get:

$$\begin{aligned} (Ts_\alpha T^{-1})(Tx) &= Ts_\alpha x, \\ &= Tx. \end{aligned}$$

Thus $Ts_\alpha T^{-1}$ fixes $H_{T\alpha}$ pointwise and since $Ts_\alpha T^{-1}$ sends $T\alpha$ to its negative:

$$\begin{aligned} (Ts_\alpha T^{-1})(T\alpha) &= Ts_\alpha \alpha, \\ &= -T\alpha. \end{aligned}$$

We see that $Ts_\alpha T^{-1} = s_{T\alpha}$. □

Given a reflection $s_\alpha \in W$ this determines a hyperplane H_α, as we have already stated, and a line $L_\alpha = \mathbb{R}\alpha$ which is orthogonal to the hyperplane. Theorem 2.1.6 implies that W permutes the set of all such lines.

2.2 Roots

2.2.1 Root Systems

The study of finite reflection groups is based around the theory of root systems and many recent developments for complex reflection groups have been to establish similar results for such groups. We will explore these developments in Chapters 4 and 6, here we build up the basic definitions and results for real reflection groups.

Definition 2.2.1. Consider a finite set Φ of nonzero vectors in V that satisfy:

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\} \quad \forall \alpha \in \Phi$,

2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$.

Then let W be the group generated by all reflections s_α such that $\alpha \in \Phi$. We call Φ a *root system* with associated reflection group W and the elements of Φ, *roots*.

The associated reflection group W is finite since each element of W fixes pointwise the orthogonal complement of the subspace spanned by Φ, as each s_α for $\alpha \in \Phi$ does. Thus, only the identity can fix all elements of Φ and the natural homomorphism of W into the symmetric group on Φ has a trivial kernel, hence W must be finite. Our choice of root system, Φ, with corresponding reflections which generate a particular reflection group need not be unique, and indeed any finite reflection group can be generated by reflections corresponding to vectors in some Φ. The aim of Chapters 2 and 3 of this project is to arrive at the classification of finite reflection groups, and since we have that all such groups are generated by Φ one would be forgiven for thinking that perhaps we are done (or at least mostly). However, the size of Φ could be very large in comparison to the dimension of the vector space. For example if $W = D_{2n}$ (the dihedral group) then we could have $|\Phi| = |W| = 2n$ but $\dim V = 2$. So it seems this may not the best way of classifying the groups, in fact we want to find a smaller, linearly independent, set from which Φ can be constructed.

This is what we later call a *simple system*.

2.2.2 Positive Systems

As seen above a root system can be very large and so we now look for useful subsets of these.

Definition 2.2.2. A *total ordering* of a real vector space V is a transitive relation on V such that:

1. For each $u, v \in V$ exactly one of the following hold:
   - $u < v$,
   - $u = v$,
   - $v < u$.
2. For all $u, v, w \in V$ if $u < v$ then $u + w < v + w$.
3. If $u < v$ and $0 \neq \lambda \in \mathbb{R}$ then $\lambda u < \lambda v$ if $\lambda > 0$ and $\lambda v < \lambda u$ if $\lambda < 0$.

Given a total ordering of V, we say that some $v \in V$ is *positive* if $v > 0$ in the ordering. The set of positive roots, $\Pi \subset \Phi$, is closed, since, the sum of positive vectors is positive and scalar multiplication by a positive real number will still give something positive. The collection of all of these positive roots are combined to form a *positive system*, which clearly must exist.

Definition 2.2.3. Given a root system Φ, a subset, $\Pi \subset \Phi$, is called a *positive system* if it consists of all roots which are positive with respect to some total ordering of V.

We know that roots come in pairs, say $\{\alpha, -\alpha\}$, and thus we can define a *negative system* in much the same way as Definition 2.2.3. We call $-\Pi$ a *negative system*, and clearly:

$$\Phi = -\Pi \sqcup \Pi.$$

Where $\sqcup$ means the disjoint union.

2.2.3 Simple Systems

Definition 2.2.4. A *simple system*, Δ, is a linearly independent set of roots such that each element of the root system, Φ, is a non-negative or non-positive linear combination of elements of Δ. We call the elements of Δ, *simple roots*.

In other words, for each $\alpha \in \Phi$, we can construct α as a linear combination of vectors in Δ with coefficients all of the same sign. Unlike positive systems, it is not obvious that simple systems must exist. However we are not in a position to prove this yet, we will in Section 2.2.5, so until then let $\Delta = \{\alpha \in \Pi \,|\, \alpha \text{ cannot be wrriten as } \beta + \gamma \text{ for } \beta, \gamma \in \Pi\}$ which must exist, and we later show that this is indeed a base of the root system Φ.

Example 2.2.5. We once again consider D_8, the dihedral group of order 8, but highlight a positive and simple system. The positive system, Π, is made up by the bold lines ($\Pi = \{\alpha, \beta, \gamma, \delta\}$) and the corresponding simple system is denoted by the bold dotted and dashed lines ($\Delta = \{\alpha, \beta\}$).

It is easy to check that the reflections corresponding to the roots α and β do indeed generate D_8. To see this consider the vectors in the positive system as unit vectors, around the origin, i.e.

$$\alpha = (1, 0),\ \beta = (-1, 1),\ \gamma = (1, 1) \text{ and } \delta = (0, 1).$$

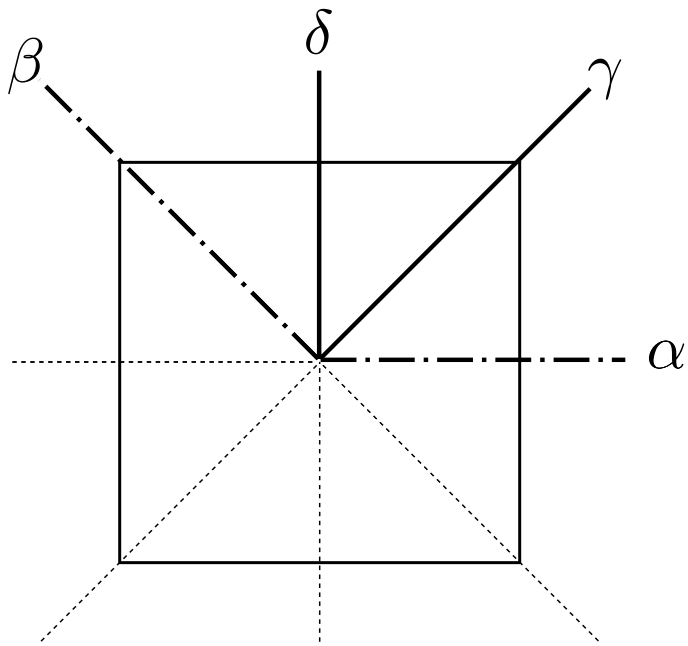


Figure 2.3: D_8, example of a positive and simple system.

Now note that,

$$\begin{aligned}\alpha + \beta &= (1,0) + (-1,1), \\ &= (0,1), \\ &= \delta.\end{aligned}$$

and

$$\begin{aligned}2\alpha + \beta &= 2(1,0) + (-1,1), \\ &= (1,1), \\ &= \gamma.\end{aligned}$$

In particular we can express γ and δ in terms of α and β with non-negative positive coefficients, and from this we can clearly also generate the corresponding negative system (multiplication by -1). Thus α and β make up the simple system.

Example 2.2.6. As in Example 2.1.5 we consider the symmetric group. This has root system $\Phi = \{e_i - e_j \mid i \neq j\}$ since clearly condition (1) of Definition 2.2.1 is met, as is the second because $s_{e_i - e_j}$ is the permutation of the standard basis of $\mathbb{R}^n$ swapping the elements e_i and e_j and $s_{e_i-e_j}\alpha = \frac{\alpha-(e_i-e_j,\alpha)(e_i-e_j)}{2}$ for all $\alpha \in \Phi$. There are multiple choices for the positive system, for example we may choose $\Pi = \{e_i - e_j \mid i < j\}$ which has corresponding simple system $\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n\}$. Note that $|\Delta| = n - 1$.

2.2.4 Angle Between Simple Roots

Note that in Example 2.2.5 and 2.2.6 the angles between the roots in the simple system are not acute, this is not an accident and this is the motivation of this subsection. We work towards the proof of Theorem 2.2.7.

Theorem 2.2.7. *Given a finite reflection group, W, with corresponding root system Φ, positive system Π and simple system Δ, then for all $\alpha, \beta \in \Delta$ such that $\alpha \neq \beta$:*

$$(\alpha, \beta) \leq 0,$$

i.e. the angle between two distinct roots in a simple system is not acute.

Before we can prove this we first require some more results. From here on we always let Φ be the corresponding root system to a finite reflection group W, with positive system Π and simple system Δ.

Definition 2.2.8. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, a vector $v \in V$ is called *vector* *positive* provided we can write

$$v = \sum_{i=1}^{n} \lambda_i \alpha_i,$$

with all $\lambda_i \geq 0$. A vector $u \in V$ is *vector negative* if and only if $-u$ is *vector positive.*

Proposition 2.2.9. *Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a simple system for a finite reflection group W and let $\alpha_i, \alpha_j \in \Delta$ such that $\alpha_i \neq \alpha_j$ and let $x, y \in \mathbb{R}$ such that $x, y > 0$ then the vector*

$$x\alpha_i - y\alpha_j,$$

is not vector positive nor vector negative.

Proof. Let α_i, α_j, x, y be as above and for a contradiction assume that $x\alpha_i - y\alpha_j$ is vector positive. Therefore,

$$x\alpha_i - y\alpha_j = \left(\sum_{\substack{p=1 \\ p \neq i}}^{n} x_p \alpha_p \right) + z\alpha_i,$$

with $x_p \geq 0$, not all 0 and $z \geq 0$. So we have two cases to consider:

<u>Case 1:</u> $z < x$

Then,

$$\begin{aligned}
x\alpha_i - z\alpha_i &= \left(\sum_{\substack{p=1\\p\neq i}}^{n} x_p\alpha_p\right) + y\alpha_j,\\
(x-z)\alpha_i &= \left(\sum_{\substack{p=1\\p\neq i}}^{n} x_p\alpha_p\right) + y\alpha_j,\\
\alpha_i &= \frac{1}{x-z}\left(\sum_{\substack{p=1\\p\neq i}}^{n} x_p\alpha_p\right) + y\alpha_j.
\end{aligned}$$

Which means that α_i is a nonnegative linear combination of $\Delta\backslash\{\alpha_i\}$ which contradicts the minimality of Δ.

<u>Case 2:</u> $z \geq x$

In which case:

$$\begin{aligned}
-y\alpha_j &= \left(\sum_{\substack{p=1\\p\neq i}}^{n} x_p\alpha_p\right) + z\alpha_i - x\alpha_i,\\
&= \left(\sum_{\substack{p=1\\p\neq i}}^{n} x_p\alpha_p\right) + (z-x)\alpha_i.
\end{aligned}$$

Now, since $z - x \geq 0$ and $\sum_{\substack{p=1 \\ p \neq i}}^{n} x_p \alpha_p \geq 0$ clearly,

$$\left(\sum_{\substack{p=1 \\ p \neq i}}^{n} x_p \alpha_p \right) + (z - x)\alpha_i \geq 0,$$

and as $y > 0$ we have that $-y\alpha_j \leq 0$, concluding that,

$$0 \geq \left(\sum_{\substack{p=1 \\ p \neq i}}^{n} x_p \alpha_p \right) + (z - x)\alpha_i \geq 0,$$

thus,

$$\left(\sum_{\substack{p=1 \\ p \neq i}}^{n} x_p \alpha_p \right) + (z - x)\alpha_i = 0.$$

Which is a contradiction, as a nonnegative linear combination of elements of Δ with at least one not equal to 0 cannot be equal to 0.

The above two cases, and contradictions, imply that $x\alpha_i - y\alpha_j$ can not be vector positive.

Now if we assume, $x\alpha_i - y\alpha_j$ is vector negative, then:

$$-(x\alpha_i - y\alpha_j) = y\alpha_j - x\alpha_i,$$

is positive and we get a contradiction as above. Thus $x\alpha_i - y\alpha_j$ is neither vector positive nor vector negative. $\square$

Proposition 2.2.10. *Let $\alpha, \beta \in \Delta$ such that $\alpha \neq \beta$, then,*

$$s_\alpha \beta \in \Pi.$$

Proof. Clearly $s_\alpha\beta \in \Phi$, where Φ is the root system for which $\Delta \subset \Phi$ so $s_\alpha\beta$ is either positive or negative but

$$s_\alpha\beta = \beta - 2(\beta, \alpha)\alpha,$$

with one positive coefficient. By Proposition 2.2.9 both coefficients must be nonnegative so $s_\alpha\beta \in \Pi$. $\square$

Proposition 2.2.11. *Consider a finite reflection group W with simple system Δ contained in the positive system Π. If $\alpha \in \Delta$ and $\beta \in \Pi$ such that $\alpha \neq \beta$ then $s_\alpha\beta \in \Pi$.*

Remark. Proposition 2.2.11 implies that $s_\alpha(\Pi\backslash\{\alpha\}) = \Pi\backslash\{\alpha\}$.

Proof of Proposition 2.2.11. Let $\beta \in \Pi$ and first assume $\alpha \in \Delta$, then $s_\alpha\beta \in \Pi$ by Proposition 2.2.10.

If $\alpha \notin \Delta$ then $\alpha = \sum_{\delta\in\Delta} c_\delta\delta$ and at least two c_δ are positive. So consider $\gamma \in \Delta$ such that $\alpha \neq \gamma$ and let $c_\gamma > 0$. Thus,

$$\begin{aligned} s_\alpha\beta &= \sum_{\delta\in\Delta} c_\delta s_\alpha\delta, \\ &= c_\gamma\gamma + \sum_{\substack{\delta\in\Delta \\ \delta\neq\gamma}} c_\delta\delta - 2\left(\sum_{\delta\in\Delta} c_\delta(\delta,\alpha)\right)\alpha. \end{aligned}$$

Finally since $s_\alpha\beta \in \Phi = -\Pi \sqcup \Pi$ it must either be positive or negative. We know $s_\alpha\beta$ has at least one positive coefficient γ and thus all the coefficients are nonnegative, hence $s_\alpha\beta \in \Pi$. $\square$

We are now in a position to prove Theorem 2.2.7.

Proof of Theorem 2.2.7. If we consider the reflection of α along β, $s_\beta\alpha$, then (via Theorem 2.1.2) we get,

$$s_\beta \alpha = \alpha - \frac{2(\alpha, \beta)}{(\beta, \beta)}\beta.$$

If $(\alpha, \beta) > 0$ then $s_\beta \alpha$ is neither positive nor negative (using $x = 1$ and $y = \frac{2(\alpha,\beta)}{(\beta,\beta)}$ in Proposition 2.2.9). A contradiction, since clearly $s_\beta \alpha \in \Phi$ and $\Phi = -\Pi \sqcup \Pi$ so $s_\beta \alpha \in -\Pi \sqcup \Pi$ and so must be either positive or negative.

Thus $(\alpha, \beta) \leq 0$ as required. □

2.2.5 Existence of Simple Systems

Theorem 2.2.7 leaves us in a position to consider the most important result in this section on roots, that is, that our $\Delta = \{\alpha \in \Pi \,|\, \alpha$ cannot be wrriten as $\beta + \gamma$ for $\beta, \gamma \in \Pi\}$ is indeed a base, i.e the existence of simple systems.

Theorem 2.2.12. *For every positive system Π in Φ, there exists a unique simple system Δ. In particular, simple systems exist.*

Proof.

Existence:

Consider a minimum subset Δ of a positive system Π (that is $\Delta \subset \Pi$) such that for all $\alpha \in \Pi$, $\alpha = \sum_{\delta \in \Delta} a_\delta \delta$ and $a_\delta \geq 0$. It is clear that such Δ exist, we now show this is linearly independent.

If we assume Δ is linearly dependent then $\sum_{\delta \in \Delta} a_\delta \delta = 0$ with not all $a_\delta = 0$ so we can rewrite this as,

$$\sum b_\beta \beta = \sum c_\gamma \gamma,$$

with $b_\beta, c_\gamma > 0$ and $\beta \neq \gamma$. So $\sum b_\beta \beta = \sum c_\gamma \gamma > 0$ then via the definition of the inner product:

$$0 \leq \left(\sum b_\beta \beta, \sum c_\gamma \gamma \right),$$

but by Theorem 2.2.7,

$$\left(\sum b_\beta \beta, \sum c_\gamma \gamma \right) \leq 0,$$

So combining the above we get that

$$\left(\sum b_\beta \beta, \sum c_\gamma \gamma \right) = 0.$$

Contradicting that $\sum b_\beta \beta = \sum c_\gamma \gamma > 0$, thus Δ is linearly independent.

Uniqueness:

If $\Delta \subset \Pi$ then Δ is the set of all roots $\alpha \in \Pi$ such that α can not be written as a linear combination with positive coefficients of elements of Π. Thus Δ is unique in Π. □

This is a nice result, and exactly the kind of thing we want, given a positive system we can find a unique simple system. It is also easy to show that any simple system is contained in a unique positive system. Which we state in the following lemma.

Lemma 2.2.13. *Given a simple system Δ, we can find a unique positive system which contains the simple system.*

Proof.

Existence:

Take a simple system Δ, which, by definition, is a linearly independent set. Then extend the set to an ordered basis of V and we can take Π to be the set of positive elements of Φ with respect to the corresponding lexicographic ordering, thus $\Delta \subset \Pi$.

Uniqueness:

Suppose Δ is a simple system which is contained in a positive system Π, i.e. $\Delta \subset \Pi$. Then clearly all roots which are nonnegative linear combinations of elements of Δ must also be in Π, and we can characterize Π uniquely as the set of all such roots. $\square$

We see from Theorem 2.2.12 and Lemma 2.2.13 that the cardinality of any simple system does not depend on Φ and this is how we define the *rank* of the group.

Definition 2.2.14. The *rank* of W is the cardinality of the associated simple system, in particular, rank of $W = \text{rk}(W) = |\Delta|$.

Example 2.2.15.

D_{2n} (the dihedral group) has rank 2 (the converse is also true i.e. if a root system Φ has rank 2 then it is a dihedral group). The proof of this is similar to the argument given to show that D_8 has a simple system containing 2 elements as in Example 2.2.5.

$S_n = \text{Sym}(n)$ (the symmetric group) has rank $n-1$. Again this is clear from the given simple system in Example 2.2.6.

Lemma 2.2.16. *Any two positive systems,* Π*, in a root system* Φ*, are conjugate under* W*.*

Proof. Let Π and Π' be positive systems, so each contain half of the roots. We prove this result by induction on $r := \text{Card}(\Pi \cap -\Pi')$.

For $r = 0$, $\Pi = \Pi'$ and we are done.

Now consider $r \geq 1$, clearly the corresponding simple system Δ of Π cannot be completely contained in Π' so we consider an element, namely $\alpha \in \Delta$ such that $\alpha \in -\Pi'$. Proposition 2.2.11 then says that $\mathrm{Card}(s_\alpha \Pi \cap -\Pi') = r - 1$ and so induction implied to $s_\alpha \Pi$ and Π' gives us an element $w \in W$ in which $w(s_\alpha \Pi) = \Pi'$, and so Π and Π' are conjugate. $\square$

2.3 Presentation of Reflection Groups

The aim of this section is to get a presentation of W as an abstract group and we state the main theorem that we will work towards.

Theorem 2.3.1. *For a fixed simple system Δ in Φ, W is generated by the set $S := \{s_\alpha \,|\, \alpha \in \Delta\}$, subject only to the relation, $(s_\alpha s_\beta)^{m_{\alpha\beta}} = 1$ for $\alpha, \beta \in \Delta$ and where $m_{\alpha\beta}$ denotes the order of $s_\alpha s_\beta$ in W.*

The order of $s_\alpha s_\beta$ is equal to 1 if $\alpha = \beta$ and is an element of the set $\{2, 3, \ldots\}$ otherwise. We can write Theorem 2.3.1 in a slightly easier way using the definition of a *simple reflection*.

Definition 2.3.2. A *simple reflection* is a reflection s_α for which $\alpha \in \Delta$.

Giving rise to an alternative form of Theorem 2.3.1.

Theorem 2.3.1: Fix a simple system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ with simple reflections s_{α_i}. For simplicity we let $s_i = s_{\alpha_i}$ and $m_{ij} = m_{\alpha_i \alpha_j}$. Then W has the following presentation:

$$W = \langle s_1, s_2, \ldots, s_n \,|\, (s_i s_j)^{m_{ij}} = 1 \rangle.$$

Indeed any such group that has a presentation of this form is called a *Coxeter group*.

It is from this theorem that we get the definition of a Coxeter system, it is defined as follows.

Definition 2.3.3. Given a finite reflection group W with a set of generators $S := \{s_\alpha \mid \alpha \in \Delta\}$ such that $(s_\alpha s_\beta)^{m_{\alpha\beta}} = 1$, (W, S) is called a *Coxeter system.*

Example 2.3.4. The dihedral group D_8 is an example of a Coxeter group, with group presentation:

$$D_8 = \langle s_1, s_2 \mid (s_1)^2 = (s_2)^2 = (s_1 s_2)^4 = 1 \rangle.$$

2.3.1 Finite Reflection Groups are Generated by Simple Reflections

Before we can prove the presentation theorem we first must show that you can actually generate W with simple reflections. This requires another definition.

Definition 2.3.5. If $\alpha \in \Phi$ then we know you can write α in terms of elements of the corresponding simple system Δ. Say, $\alpha = \sum_{\delta\in\Delta} c_\delta \delta$, we call $\sum_{\delta\in\Delta} c_\delta$ the *height* of α relative to Δ.
In particular $\mathrm{ht}(\alpha) = \sum_{\delta\in\Delta} c_\delta$.

Theorem 2.3.6. *Given a finite reflection group W with a corresponding fixed simple system Δ, then W is generated by simple reflections, that is reflections s_α such that $\alpha \in \Delta$.*

Proof. Fix a simple system Δ and let W' be the group generated by Δ, we want to show $W' = W$.

From Lemma 2.2.13 we know that Δ is uniquely contained in some positive system Π. Now if $\beta \in \Pi$ consider $W'\beta \cap \Pi$ which is a nonempty set of positive roots and choose $\gamma \in (W'\beta \cap \Pi)$ of minimum height.

Claim: $\gamma \in \Delta$.

Proof of claim. Write γ as $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$ and note that $0 < (\gamma, \gamma) = \sum_{\alpha \in \Delta} c_\alpha (\gamma, \alpha)$ which implies $(\gamma, \alpha) > 0$ for some $\alpha \in \Delta$. Now, if $\gamma = \alpha$ we are done, so say $\gamma \neq \alpha$ and consider $s_\alpha \gamma$.
We obtain $s_\alpha \gamma$ from γ by subtracting positive multiples of α and thus $\mathrm{ht}(s_\alpha \gamma) < \mathrm{ht}(\gamma)$ contradicting our choice of γ (we choose γ to have minimum height).

Thus $\gamma = \alpha$, so $\gamma \in \Delta$.

Claim: $W'\Delta = \Phi$.

Proof of claim. We know from above that any W'-orbit of a positive root meets the simple system, thus $\Pi \subseteq W'\Delta$. Whereas if β is negative then $-\beta \in \Pi$ is conjugate by some $w \in W'$ to $\alpha \in \Delta$. Then $-\beta = w\alpha$ which implies that $\beta = (ws_\alpha)\alpha$, with $ws_\alpha \in W'$. So $-\Pi \subseteq W'\Delta$, in particular $W'\Delta = \Phi$.

Claim: $W' = W$.

Proof of claim. Consider a generator s_β of W. We can write $\beta = w\alpha$ for some $w \in W'$, $\alpha \in \Delta$ via the above claim. Then Theorem 2.1.6 implies that $s_\beta = ws_\alpha w^{-1} \in W'$.

Thus, $W = W'$.

So W is generated by simple reflections. □

2.3.2 The Length Function

Definition 2.3.7. The *length*, $l(w)$, of w, relative to some simple system Δ, is the smallest l such that $w = s_{\alpha_1} \cdots s_{\alpha_l}$ where s_{α_i} are simple reflections.

The length of an element $w \in W$ is equal to 1 ($l(w) = 1$) if and only if $w = s_\alpha$. Also $l(w) = l\left(w^{-1}\right)$ since if $w = s_{\alpha_1} \cdots s_{\alpha_l}$ then $w^{-1} = s_{\alpha_l} \cdots s_{\alpha_1}$ which implies $l\left(w^{-1}\right) \leq l(w)$ and vice versa so $l(w) = l\left(w^{-1}\right)$.

Definition 2.3.8. The *number of positive roots sent to negative roots by* $w =: n(w) := \mathrm{Card}(\Pi \cap w^{-1}(-\Pi))$.

The above function is sometimes called the "n function".

Lemma 2.3.9. *Let $\alpha \in \Delta$ and $w \in W$ then:*

1. *If $w\alpha \in \Pi$ then $n(ws_\alpha) = n(w) + 1$.*
2. *If $w\alpha \in -\Pi$ then $n(ws_\alpha) = n(w) - 1$.*

Proof. Let $\Pi(w) := \Pi \cap w^{-1}(-\Pi)$, so that $n(w) = \mathrm{Card}\, \Pi(w)$. If $w\alpha \in \Pi$ then via Proposition 2.2.11 $\Pi(ws_\alpha)$ is the disjoint union of $s_\alpha \Pi(w)$ and $\{\alpha\}$ and thus $n(ws_\alpha) = n(w) + 1$.
If $w\alpha \in -\Pi$ then the same argument gives $s_\alpha \Pi(ws_\alpha) = \Pi(w) \backslash \{\alpha\}$ but α is not in $\Pi(w)$ thus $n(ws_\alpha) = n(w) - 1$ □

Lemma 2.3.10. *With $n(w)$ and $l(w)$ defined as above,*

$$n(w) = l(w).$$

Proof. Let $w = s_1 \cdots s_r$ then $n(w) \leq l(w)$ since we can get to the expression for w in r steps, and so the the number of positive roots sent to negative roots can go up by at most 1 at each step and thus $n(w) \leq l(w)$.

Now assume $n(w) < l(w) = r$ so we can write $w = s_{\alpha_1} \cdots s_{\alpha_r}$. Since $n(w) < r$ if we repeatedly apply Lemma 2.3.9 we find a $j \leq r$ such that $(s_{\alpha_1} \cdots s_{\alpha_{j-1}})\alpha_j < 0$, but as $\alpha_j > 0$ there exists $i < j$ such that $s_{\alpha_i}(s_{\alpha_{i+1}} \cdots s_{\alpha_{j-1}})\alpha_j < 0$ and $(s_{\alpha_{i+1}} \cdots\ s_{\alpha_{j-1}})\alpha_j > 0$. Then Proposition 2.2.11 implies $\alpha_i = (s_{\alpha_{i+1}} \cdots s_{\alpha_{j-1}})\alpha_j$ and Theorem 2.1.6 gives,

$$(s_{\alpha_{i+1}} \cdots s_{\alpha_{j-1}})s_{\alpha_j}(s_{\alpha_{j-1}} \cdots s_{\alpha_{i+1}}) = s_{\alpha_i},$$

and thus we can write $w = s_{\alpha_1} \cdots \hat{s_{\alpha_i}} \cdots \hat{s_{\alpha_j}} \cdots s_{\alpha_r}$ where $\hat{s_{\alpha_i}}$ means that s_{α_i} is omitted, so $l(w) = r - 2$ which contradicts that $l(w) = r$. Thus $n(w) \geq l(w)$.

Hence $l(w) = n(w)$. □

Remark. So as $l(w) = n(w)$ we get from Lemma 2.3.9 that $l(ws_\alpha) = l(w) \pm 1$ depending on whether $w\alpha \in \Pi$ or $-\Pi$.

Definition 2.3.11. If $W \ni w = s_{\alpha_1} \cdots s_{\alpha_l}$ then we call each w_j such that $w_j = s_{\alpha_1} \cdots s_{\alpha_j}$ with $0 \leq j \leq k$, a *partial element* of w.

If $j = 0$ then $w_j = w_0$ has no factors and so we take by convention that $w_0 = 1$.

Notation. If s_α and s_β are fixed simple reflections then by $(s_\alpha s_\beta \cdots)_k$ we denote the element $s_\alpha s_\beta s_\alpha s_\beta \cdots$ having a total of k factors (starting with s_α). Similarly $(\cdots s_\alpha s_\beta)_k$ denotes the element $\cdots s_\alpha s_\beta$ having a total of k factors (ending with s_β).

Example 2.3.12. $(s_\alpha s_\beta \cdots)_5 = s_\alpha s_\beta s_\alpha s_\beta s_\alpha$.
$(\cdots s_\alpha s_\beta)_5 = s_\beta s_\alpha s_\beta s_\alpha s_\beta$.

Proposition 2.3.13. *If s_α and s_β are simple reflections in W and $1 \leq p \leq m_{\alpha\beta}$ then*

$$(\cdots s_\alpha s_\beta)_{p-1}\alpha \in \Pi.$$

Proof. By the definition of a simple reflection we have $\alpha, \beta \in \Delta$. If $\alpha = \beta$ then $m_{\alpha\beta} = 1 = p$ and so the result is trivial.

So let $\alpha \neq \beta$ and assume the result is false.
Take the smallest value of p for which $(\cdots s_\alpha s_\beta)_{p-1}\alpha \in -\Pi$. From the above we know $p > 1$ and we consider two cases:

<u>Case 1:</u> p is even.

In which case

$$\begin{aligned}(\cdots s_\alpha s_\beta)_{p-1}\alpha &= (s_\beta \cdots s_\alpha s_\beta)_{p-1}\alpha, \\ &= s_\beta(\cdots s_\alpha s_\beta)_{p-2}\alpha \in -\Pi,\end{aligned}$$

and $(\cdots s_\alpha s_\beta)_{p-2}\alpha \in \Pi$ by the minimality of p. Then via Proposition 2.2.11 $(\cdots s_\alpha s_\beta)_{p-2}\alpha = \beta$.
Therefore $s_\beta = (\cdots s_\alpha s_\beta)_{p-2} s_\alpha (\cdots s_\alpha s_\beta)_{p-2}^{-1}$ we get that $s_\beta(\cdots s_\alpha s_\beta)_{p-2} = (\cdots s_\alpha s_\beta)_{p-2} s_\alpha$ which is equivalent to $(s_\beta \cdots s_\alpha s_\beta)_{p-1} = (s_\alpha \cdots s_\beta s_\alpha)_{p-1}$. But then we have that $(s_\alpha s_\beta \cdots)_{2p-2} = (s_\alpha s_\beta)^{p-1} = 1$.

Contradicting that $s_\alpha s_\beta$ has order $m_{\alpha\beta}$.

<u>Case 2:</u> p is odd.

The proof for p being odd is analogous to the case for even p. $\square$

Proposition 2.3.14. *Let Δ be a simple system with simple reflections $S = \{s_1, \ldots, s_n\}$. If $w \in W$, α and β are fixed and $l(ws_\alpha) = l(ws_\beta) = l(w) - 1$,* *then*

$$l(w(\cdots s_\alpha s_\beta)_p) = l(w) - p,$$

if $0 \leq p \leq m_{\alpha\beta}$.

Proof. This is proved by induction on p.
If $p = 0$ or $p = 1$ the result is trivial.

Suppose $p \geq 2$ and

$$l(w(\cdots s_\alpha s_\beta)_{p-1}) = l(w) - (p-1), \text{ if } 0 \leq p \leq m_{\alpha\beta},$$

holds. By Lemmas 2.3.9 and 2.3.10 we know that $w\alpha$ and $w\beta$ are in $-\Pi$ and by Proposition 2.3.13 the root $(\cdots s_\alpha s_\beta)_{p-1}\alpha$ is positive, and so it is of the form $a\alpha + b\beta$ with $a, b \geq 0$ and not both zero. So

$$\begin{aligned} w(\cdots s_\alpha s_\beta)_{p-1}\alpha &= w(a\alpha + b\beta), \\ &= aw\alpha + bw\beta \in -\Pi, \end{aligned}$$

and so

$$\begin{aligned} l(w(\cdots s_\beta s_\alpha)_p) &= l(w(\cdots s_\alpha s_\beta)_{p-1} s_\alpha), \\ &= l(w(\cdots s_\alpha s_\beta)_{p-1}) - 1, \\ &= l(w) - (p-1) - 1, \\ &= l(w) - p, \end{aligned}$$

again by Lemmas 2.3.9 and 2.3.10, Proposition 2.3.13 and the induction hypothesis.

Similarly $l(w(\cdots s_\alpha s_\beta)_p) = l(w) - p$. $\square$

2.3.3 Proof of the Presentation of Reflection Groups

Now using Proposition 2.3.14 we can prove Theorem 2.3.1.

Proof of Theorem 2.3.1. Let $W \ni w = s_{\alpha_1} \cdots s_{\alpha_k}$ where s_{α_i} are simple reflections (for ease of notation in this proof we equivalently write $w = s_1 \cdots s_k$). Suppose that u is the maximum length of any partial element of W. Then we can write w in another way, letting $j = i + 1$,

$$\begin{aligned} w &= s_1 \cdots s_k, \\ &= s_1 \cdots s_{i-1} s_i s_j s_{j+1} \cdots s_k, \\ &= w_1 s_i s_j w_2, \end{aligned}$$

where $l(w_1 s_i) = u$ and every partial element of w_1 has length less than u. Let $m = m_{ij}$ and $w' = w_1(s_j s_i \cdots)_{2m-2} w_2$. Then note that $w = w'$, as elements of the group W and, with the exception of $w_1 s_i$, all partial elements of w, coincide with the partial elements of w'. In place of $w_1 s_i$, w' has the following partial elements:

$$w_1 s_j,\ w_i s_j s_i,\ \ldots,\ w_1(s_j s_i \cdots)_{2m-3}.$$

Now let $v = w_1 s_i$ and since $s_i^2 = 1$, we see that the above partial elements of w' coincide with the elements $v(s_i s_j \cdots)_p$ for $2 \leq p \leq 2m-2$. This is because,

$$\begin{aligned} v(s_i s_j \cdots)_p &= w_1 s_i (s_i s_j \cdots)_p, \\ &= w_1 s_i^2 (s_j s_i \cdots)_{p-1}, \\ &= w_1 (s_j s_i \cdots)_{p-1}. \end{aligned}$$

Now as $l(v) = u$ and thus $l(v)$ is maximal we get,

$$l(vs_i) = l(vs_j) = l(v) - 1.$$

If $2 \leq p \leq m$ then $l(v(s_i s_j \cdots)_p) < u$ by Proposition 2.3.14.
If $m < p < 2m - 2$ then rearranging gives $2 \leq 2m - p < m$ and again by Proposition 2.3.14,

$$l(v(s_i s_j \cdots)_p) = l(v(s_j s_i \cdots)_{2m-p}) < u.$$

So now by applying $(s_i s_j)^{m_{\alpha\beta}}$ we have replaced w by w' all whose partial elements have length less than or equal to u and having one fewer partial element of length equal to u.

We now repeat the above procedure until we arrive at $1 = 1$, which we must get since we are dealing with finite sets, and we are done. $\square$

Chapter 3

Classification of Real Reflection Groups

> Modern algebra does not seem quite so terrifying when expressed in these geometrical terms!
>
> G. de B. Robinson [20, pg. 94]

3.1 Coxeter Graph

Theorem 2.3.1 and Lemma 2.2.16 imply that a finite reflection group W is determined, up to isomorphism, by the set of integers $m_{\alpha\beta}$ with $\alpha, \beta \in \Delta$. We can express this information in terms of a graph, namely the *Coxeter graph* of W, and the aim of this chapter is to classify all possible finite reflection groups in terms of their Coxeter graph.

Definition 3.1.1. A graph Γ, the *Coxeter graph* of W, has its vertex set in one-to-one correspondence with $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and has an edge between

a pair of vertices whenever $m_{\alpha_i\alpha_j} \geq 3$ (for simplicity we let $m_{ij} = m_{\alpha_i\alpha_j}$). We label such an edge with m_{ij}.
If a pair of vertices are not connected then this means $m_{ij} = 2$, and by definition $m_{ii} = 1$.

Example 3.1.2. If we take $W = D_8$, then we know that D_8 has two roots in its simple system, from Examples 2.1.4 and 2.2.5, say α and β, and $m_{\alpha\beta} = 4$. So it has the following Coxeter graph.

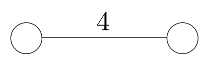


Figure 3.1: Coxeter Graph of D_8.

A Coxeter graph determines W up to isomorphism and we now reserve Γ to represent such a graph. Lemma 2.2.16 says that positive systems are conjugate and since each positive system contains a unique simple system (Theorem 2.2.12), we have that simple systems are conjugate and thus the graph does not depend on the choice of Δ.

Definition 3.1.3. Given any Coxeter graph Γ, we define a *subgraph*, Γ', of Γ by either removing an edge of order 3 and merging the respective vertices or by lowering the order of any edger greater than 3.

We can apply the above defined process recursively to Coxeter graphs to get a series of different, and smaller, subgraphs from the same Γ.

Example 3.1.4. The graphs in Figure 3.2 are examples of a graph and a subgraph.

Definition 3.1.5. We say the Coxeter system (W, S), as defined in Definition 2.3.3, is *irreducible* if the Coxeter graph Γ is connected.

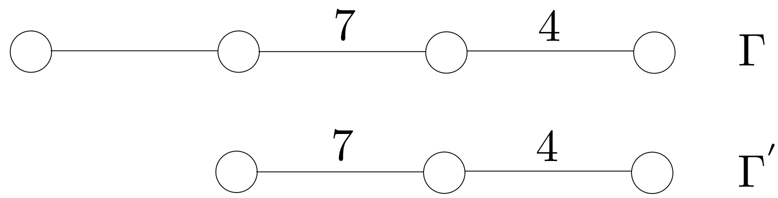


Figure 3.2: Γ' is a subgraph of Γ

We call S, the set of simple reflections of W, *irreducible* in this case.

Before we proceed with classifying the finite reflection groups we argue that we need only consider the connected Coxeter graphs.

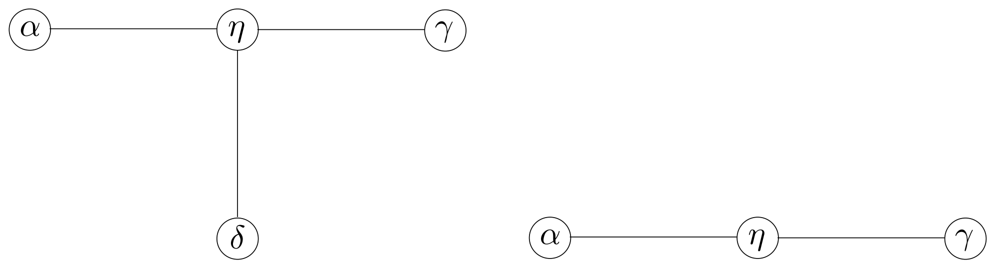
Theorem 3.1.6. *Let (W, S) have a Coxeter graph Γ which contains connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ where $S_1, \ldots, S_r$ are the corresponding subsets of S. Then $W = W_{S_1} \times W_{S_2} \times \cdots \times W_{S_r}$ where the W_{S_i} are parabolic subgroups (subgroups of W generated by all the simple reflection from S_i) and each (W_{S_i}, S_i) is irreducible.*

Proof. We prove this by induction on r.
If $r = 1$ then Γ has only one connected component, i.e. Γ is connected, thus by our definition (W, S) is irreducible.

Now assume $r \geq 2$ and the result holds for $r - 1$.
Elements of S_i commute with elements of S_j (for $i \neq j$) and thus the parabolic subgroups are normal in W, i.e. $W_{S_i} \triangleleft W$ for $i = 1, \ldots, r$. So $S \subseteq W_{S_1} \times W_{S_2} \times \cdots \times W_{S_r}$ and so this must be all of W. By induction $W_{S \setminus S_i}$ is the direct product of W_{S_j} and $W_{S_i} \cap W_{S \setminus S_i} = \{e\}$, where e is the identity of the group. So the product is a direct product and we are done. $\square$

Definition 3.1.7. A vertex in a Coxeter graph Γ is a *branch point* if it has degree greater than or equal to 3. A Coxeter graph is a *chain* if it is a tree (graph with no cycles) and has no branch points.

Example 3.1.8. Figure 3.3 is an example of a graph with a branch point, and a graph which is a chain.



(a) η is a branch point. (b) The above graph is a chain.

Figure 3.3: Example of a graph with a branch point and which is a chain.

Now for each Coxeter graph, Γ, we can associate with it a bilinear form β, we also use the fact that bilinear forms have a matrix representation.

3.2 Bilinear Form of Coxeter Graph

Definition 3.2.1. Let the underlying Euclidean vector space, V, have basis, $\{e_1, e_2, \ldots, e_n\}$ then we define a symmetric bilinear form in the following way. To each Coxeter graph Γ, which has a vertex set S, where $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, i.e. $|S| = n$, let:

$$\begin{aligned} \beta : V \times V &\longrightarrow \mathbb{R} \\ (e_i, e_j) &\longmapsto -\cos\left(\frac{\pi}{m_{ij}}\right) \end{aligned}$$

where $m_{ij} = m_{\alpha_i \alpha_j}$.

We let $q(x) = \beta(x,x)$ and restrict $q(x)$ to when it is positive definite (i.e. $q(x) > 0$). Also let $q_{ij} = \beta(e_i, e_j)$ (so $q_{11} = \beta(e_1, e_1) = 1$) and define

$$\beta(x,y) = \sum_{i,j} q_{ij} x_i y_j,$$

for $x, y \in V$.

Definition 3.2.2. Now, with the bilinear form defined as above we can arrive at its $n \times n$ matrix representation, A, by setting:

$$a_{i,j} = q_{ij} = B(e_i, e_j).$$

The matrix A is clearly symmetric due to the symmetric condition on the bilinear form. We refer to this matrix as the *associated matrix* to the Coxeter graph.

Although we have that $a_{i,j} = q_{ij}$, we use both throughout this chapter, the former when referring to the associated matrix and the latter for the Coxeter graphs bilinear form. Hopefully this will make it clear which associated form we refer to.

Example 3.2.3. We find the associated matrix of the Coxeter graph given in Figure 3.4.

Reading from the graph we find the orders of the roots:

$$m_{\alpha\alpha} = 1,\ m_{\alpha\beta} = 4,\ m_{\alpha\gamma} = 3,\ m_{\beta\alpha} = 4,\ m_{\beta\beta} = 1,$$

$$m_{\beta\gamma} = 7,\ m_{\gamma\alpha} = 3,\ m_{\gamma\beta} = 7,\ m_{\gamma\gamma} = 1.$$

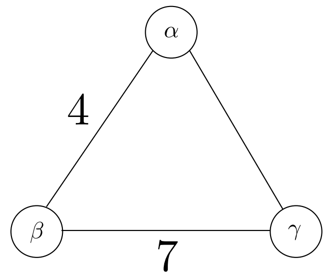


Figure 3.4: Example Coxeter graph

Then the associated matrix is:

$$A = \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 1 & -\cos\left(\frac{\pi}{7}\right) \\ -\frac{1}{2} & -\cos\left(\frac{\pi}{7}\right) & 1 \end{pmatrix}.$$

Definition 3.2.4. Given a Coxeter graph Γ with associated $n \times n$ symmetric matrix A and $x \in \mathbb{R}^n$, we call A *positive definite* if $x^T A x > 0$ for all $x \neq 0$ and *positive semidefinite* if $x^T A x \geq 0$ for all $x \neq 0$.

Similarly we can think of the positive definiteness of a bilinear form since,

$$\beta(x, y) = x^T A y.$$

When the Coxeter graph Γ is associated to a finite reflection group then it's associated matrix (and bilinear form) is positive definite. This is since it represents the usual Euclidean inner product relative to the simple system Δ. The converse is also true, but requires a more complex argument, which we include as the following lemma.

Lemma 3.2.5. *If a bilinear form is positive definite, then reflection group, W, associated to such a form is finite.*

Proof. Note that by V^* we denote the dual space of V, i.e. the set of all linear functionals. Also, a fundamental domain, for a group W, is a subset U of the vector space V such that $U \cap wU = \varnothing$ for all $w \in W \backslash \{1\}$.

For $v \in V$ consider the linear map, φ, from $v \in V$ to $fv \in V^*$ via $(fv)u = \beta(v, u)$ for all $u \in V$. This map is an isomorphism of vector spaces and since, for $w \in W$ and for all $u, v \in V$

$$(f(wv))u = \beta(wv, u) = \beta(v, w^{-1}u) = (fv)(w^{-1}u) = (w(fv))u,$$

we get that $fw = wf$ for all $w \in W$. Thus φ is an isomorphism of W-modules.

Now let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and the collection of simple reflections be S (so $|S| = n$). Then

$$U := \bigcap_{s \in S} \{g \in V^* \mid g(\alpha_s) \geq 0\},$$

is a fundamental domain of W in V^*, to check this see [7, pg. 455] and thus $f^{-1}U$ is a fundamental domain for W in V. Finally, as $f^{-1}U$ is nonempty and intersects the halfspaces $\{v \in V \mid \beta(v, \alpha_s) \geq 0\}$, we get that $f^{-1}U$ intersects the unit ball with volume λ, where $\lambda > 0$. Then $\bigcup_{w \in W} w(f^{-1}U)$ has volume $\lambda|W|$ and is contained in the unit ball, and so the value of $\lambda|W|$ must be bounded by the volume of the unit ball. Hence $|W|$ is finite, as required. □

We are now in a position to start proving some results with regards to classifying the real reflection groups.

Lemma 3.2.6. *If a Coxeter graph Γ has an associated positive definite bilinear form then any subgraph, Γ', also has an associated bilinear form that* *is positive definite.*

Proof. Given a Coxeter graph Γ with n vertices, which has associated matrix A, consider a nontrivial subgraph Γ' with associated matrix A' such that A' is a $k \times k$ matrix for some $k \leq n$. By the definition of Γ' the edge labels must satisfy $m'_{ij} \leq m_{ij}$ where $a'_{ij} = -\cos\left(\frac{\pi}{m_{ij}}\right) \geq -\cos\left(\frac{\pi}{m_{ij}}\right) = a_{ij}$.

Now assume that A' is not positive definite, i.e. there exists a nonzero $x \in \mathbb{R}^k$ such that $x^T A' x \leq 0$. Applying the quadratic form associated with the matrix A to the vector with coordinates $(|x_1|, |x_2|, \ldots, |x_k|, 0, 0, \ldots, 0) \in \mathbb{R}^n$ we get:

$$0 \leq \sum_{i,j \leq k} a_{ij}|x_i||x_j| \leq \sum_{i,j \leq k} a'_{ij}|x_i||x_j| \leq \sum_{i,j \leq k} a'_{ij} x_i x_j \leq 0,$$

where the penultimate inequality comes from the fact that $a'_{ij} \leq 0$ for $i \neq j$. Thus equality holds throughout and we get,

$$\sum_{i,j \leq k} a_{ij}|x_i||x_j| = \sum_{i,j \leq k} a'_{ij}|x_i||x_j|.$$

This implies that $a_{ij} = a'_{ij}$ for $1 \leq i, j \leq k$, and we must have $k < n$. But the first equality shows we have a null vector for A which forces $k = n$, contradicting the fact that Γ' was a proper subgraph. So A' is positive definite. $\square$

3.3 Restriction of Possible Graphs

We now build up a series of results about Coxeter graphs with positive definite bilinear form which limit our choices for such graphs. We will conclude that there are in fact only ten such types of graphs, each of which correspond to a finite irreducible Coxeter system, and we will have thus classified them. Our argument follows in a similar vein to [31].

From here onwards, namely in all of the results and their proofs, we assume that Γ is a connected positive definite Coxeter graph.

Lemma 3.3.1. Γ *is a tree (i.e. a connected graph with no cycles).*

Proof. Let Γ have m vertices, and assume it contains a cycle and denote this by $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}\alpha_{i_1}$ for $k \geq 3$ (as in Figure 3.5).

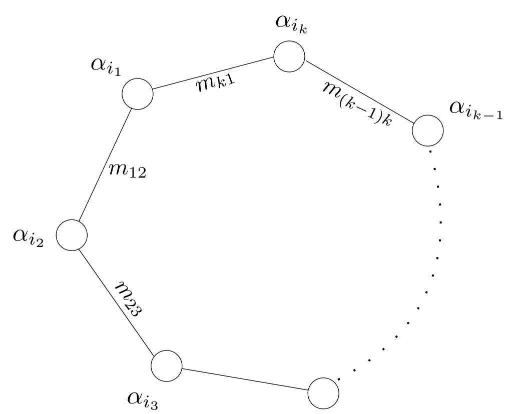


Figure 3.5

Consider the element formed from basis vectors in the following way, $e_{i_1} + e_{i_2} + \cdots + e_{i_k}$ and let this equal x. Now since $q_{ii} = 1$ and $q_{ij} = q_{ji}$ we get the following relationship,

$$q(x) = k + 2\sum_{\substack{i,j=1 \\ i \neq j}}^{k} q_{ij}, \tag{3.1}$$

As there is an edge between each consecutive term in the cycle, i.e. an edge between α_{i_p} and $\alpha_{i_{p+1}}$ for $1 \leq p \leq k$ where $\alpha_{i_{k+1}} = \alpha_{i_1}$, we must have that $m_{12}, m_{23}, \ldots, m_{(k-1)k}, m_{k1} \geq 3$ (by our definition of a Coxeter graph. Which forces

$$q_{ij} \leq -\cos\left(\frac{\pi}{3}\right), \quad \text{for } 1 \leq i, j \leq k \text{ and } i \neq j, \tag{3.2}$$

by the definition of q_{ij}.

Combining equations (3.1) and (3.2) we get $q(x) \leq k - k = 0$, contradicting the positive definiteness of $q(x)$, so Γ does not contain a cycle. Thus Γ is a tree. □

Lemma 3.3.2. *Let $\{e_1, \dots, e_n\}$ be a basis of V and β the associated bilinear form to Γ then, for a fixed i, we have that,*

$$\sum_{j \neq i} q_{ij}^2 < 1,$$

where q_{ij} is as in Definition 3.2.1, namely $q_{ij} = \beta(e_i, e_j)$.

Proof. For this proof consider a fixed i and let $J = \{j \mid j \neq i,\ q_{ij} \neq 0 \text{ and } 1 \leq j \leq n\}$.

Claim: The set $\{e_j\}_{j \in J}$ is an orthonormal basis of $U = \bigoplus_{j \in J} \mathbb{R}e_j$.

Proof of Claim. Since $q_{jj} = 1$ by definition we just need to show that $q_{jk} = 0$ for all $j \neq k \in J$. So for a contradiction assume that $q_{jk} \neq 0$, then there must be an edge between a vertex α_j and α_k in Γ, but by our definition of J we have that $q_{ij} \neq 0$ and $q_{ik} \neq 0$ again meaning there must be an edge between α_i, α_j and α_i, α_k, if we consider these three edges we obviously get a cycle (Figure 3.6) contradicting Lemma 3.3.1, i.e. $q_{ik} = 0$ as required.

Finally, we let d denote the distance from e_i to U and $\tilde{e}_i$ be the projection of e_i on U. Then from the claim we get that $\tilde{e}_i = \sum_{j \in J} q_{ij} e_j$ and since $q_{ij} = 0$

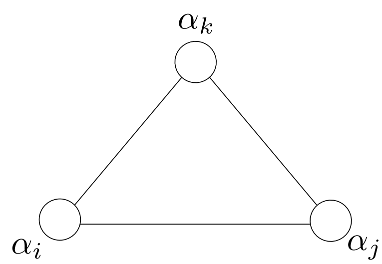


Figure 3.6

if $j \neq i$ and $j \in J$ we conclude that, $\sum_{j \in J} q_{ij} e_j = \sum_{j \neq i} q_{ij} e_j$. Hence,

$$\tilde{e_i} = \sum_{j \neq i} q_{ij} e_j. \tag{3.3}$$

Then by construction of d and $\tilde{e_i}$ and via Pythagoras' Theorem we get:

$$\|e_i\|^2 = \|\tilde{e_i}\|^2 + d^2. \tag{3.4}$$

Equations (3.3) and (3.4) give $1 = d^2 + \sum_{j \neq i} q_{ij}^2$. Then as $d^2 > 0$ we conclude that

$$\sum_{j \neq i} q_{ij}^2 < 1.$$

□

We now state and prove a theorem which we will use repeatedly, along with Lemma 3.2.6, in order to classify the Coxeter graphs. It restricts our choice of Γ further than it must be a tree.

Theorem 3.3.3. *For any Coxeter graph Γ, corresponding to a finite group, the following must hold:*

1. *The degree of a vertex in Γ is at most 3.*

2. *A vertex has degree exactly 3 only when all the edges from it have order 3.*

3. *At most one edge has order greater than or equal to 4.*

4. *There exists an edge of order greater than or equal to 6 if there are only 2 vertices.*

Proof. The proof easily follows from the fact that $q_{ij} = -\cos\left(\frac{\pi}{m_{ij}}\right)$, where $i = \alpha_i$ and $j = \alpha_j$ for some α_i and α_j in the vertex set of Γ.

1. If there is an edge between i and j in Γ, then we must have $m_{ij} \geq 3$ and thus $q_{ij} \leq -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$. So we have that $q_{ij}^2 \geq \frac{1}{4}$. Then by Lemma 3.3.2 ($\sum_{j \neq i} q_{ij}^2 < 1$) there can be at most 3 such edges.

2. Let's assume not, so there is a vertex of Γ, say a which has degree 3 but only two of these edges (corresponding to vertices i and j say) have order 3 and the last one (vertex k) has order > 3. Then we get that,

$$q_{ai} = -\frac{1}{2}, \quad q_{aj} = -\frac{1}{2}, \quad q_{ak} \leq -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

   So we sum the squares of these and compare with Lemma 3.3.2, i.e.

$$\sum_{m \neq n} q_{nm}^2 = q_{ai}^2 + q_{aj}^2 + q_{ak}^2 \geq \frac{1}{4} + \frac{1}{4} + \frac{2}{4} \geq 1,$$

   a contradiction. So we must have that a vertex has degree exactly 3 only when all the edges from it have order 3.

3. Again let's assume not, i.e. there are two edges of order ≥ 4 in the graph Γ. So there exists a subgraph Γ' of Γ with a vertex (say a) which belongs to 2 edges of order ≥ 4 (corresponding to vertices i and j). We

then have that

$$q_{ai} \leq -\frac{\sqrt{2}}{2},\ q_{aj} \leq -\frac{\sqrt{2}}{2} \quad \text{and so} \quad q_{ai}^2 \geq \frac{2}{4},\ q_{aj}^2 \geq \frac{2}{4},$$

thus $\sum_{j \neq i} q_{ij}^2 \geq 1$ contradicting Lemma 3.3.2, so there can be at most one edge of order ≥ 4.

4. Assume that there is an edge of order ≥ 6 in the graph Γ (so there must be at least two vertices, i and j) then we have that $q_{ij} \leq -\frac{\sqrt{3}}{2}$ and so $q_{ij}^2 \geq \frac{3}{4}$. Via Lemma 3.3.2 we know that if we add another edge (from vertex i to say k) to the graph then we must have $q_{ik}^2 < \frac{1}{4}$ as not to form a contradiction. But the smallest possible value is $q_{ik}^2 = \frac{1}{4}$, since this corresponding to an edge of minimum order (namely 3), so we can not find such an edges. Thus there are exactly 2 vertices in the graph Γ. □

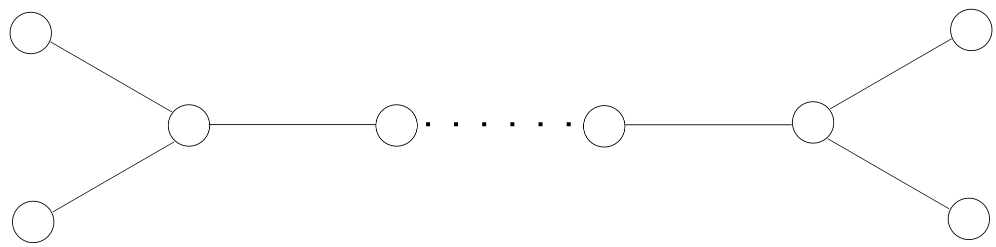
Proposition 3.3.4. *There are only two possible types of positive definite Coxeter graphs* Γ*,*

1. Γ *is a chain and has at most one edge of order greater than or equal to 4.*

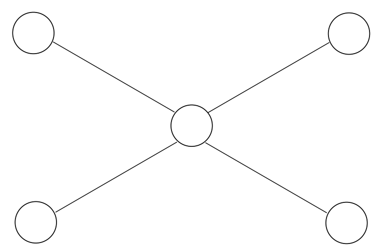
2. Γ *contains a unique branch point and all edges are of order 3.*

Proof. Assume that Γ is not one of the above graphs, in which case it must either be, or have a subgraph of, one of the following three graphs:

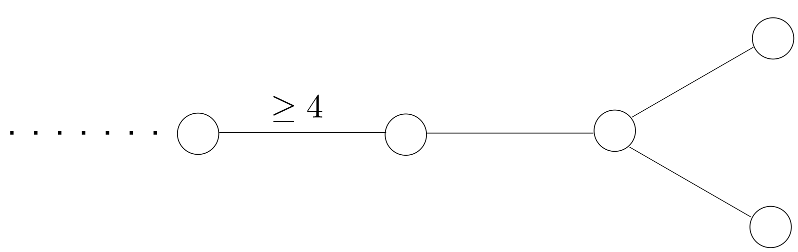
1. Two branch points separated by edges of order 3.



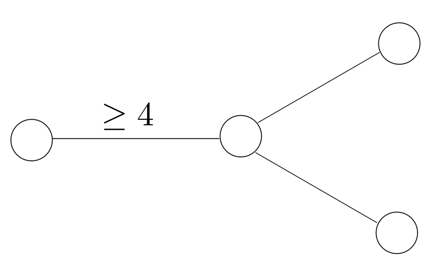
Which contradicts part (1) of Theorem 3.3.3 since this would contain as a subgraph:



2. A branch point and an edge of order greater than or equal to 4 separated by edges of order 3.

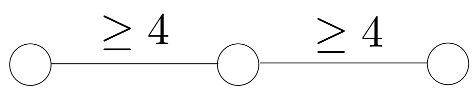


Which contradicts part (2) of Theorem 3.3.3 since this would contain as a subgraph:



3. A chain with two edges of order greater than or equal to 4.

Which contradicts part (3) of Theorem 3.3.3 since this would contain as a subgraph:



Thus there are only two possible types of positive definite Coxeter graphs, either,

1. a chain with at most one edge of order greater than or equal to 4, or
2. a unique branch point with all edges of order 3. $\square$

We now consider each of the two types of positive definite Coxeter graphs in turn to again reduce the possible graphs.

3.4 Γ a chain and has at most one edge of order ≥ 4

For all of this section we let Γ have vertex set $\{\alpha_1, \alpha_2, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_l\}$, which we will simplify as $\{1, 2, \ldots, i, i+1, \ldots, l\}$ and let $m_{\alpha_i \alpha_{i+1}} = m_{i(i+1)} = m$ and thus restrict our attention to graphs of the form in Figure 3.7.

Lemma 3.4.1. *Let* $1, \ldots, l$ *be the vertices of* Γ *(as in Figure 3.7) such that the edge between* $j(j+1)$ *has order 3 for* $1 \leq j \leq l-1$*. Then for a vector* $v \in V$ *of the form* $v = e_1 + 2e_2 + 3e_3 + \cdots + le_l$ *we have* $\|v\|^2 = \frac{1}{2}l(l+1)$.

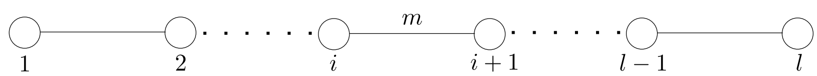


Figure 3.7: A chain with at most one edge of order ≥ 4

Proof. Since the edges $j(j+1)$ have order 3 the bilinear form gives $\beta(e_j, e_j) = 1$, $\beta(e_j, e_{j+1}) = -\frac{1}{2}$, and $\beta(e_j, e_k) = 0$ if $k \neq j \pm 1$. So

$$
\begin{aligned}
(v, v) &= \sum_{j=1}^{l} j^2 - 2\sum_{j=1}^{l-1} \frac{1}{2} j(j+1), \\
&= l^2 - \sum_{j=1}^{l-1} j, \\
&= l^2 - \frac{1}{2} l(l-1), \\
&= \frac{1}{2} l(l+1). \qquad \square
\end{aligned}
$$

Again we consider Γ from Figure 3.7, if $l = 2$ then for $m \geq 3$ we have the dihedral group, D_{2n}, as shown in Figure 3.8.

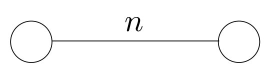


Figure 3.8: Coxeter graph of Dihedral group D_{2n}

So now consider $l \geq 3$, then via part 4 of Theorem 3.3.3 we have that $3 \leq m \leq 5$. Which gives rise to the following proposition.

Proposition 3.4.2. *For $l \geq 3$ and $3 \leq m \leq 5$ the only possible positive definite graphs are the ones given in Figure 3.9 (with corresponding group type next to each).*

Proof. $A_n \, (n \geq 1)$. We clearly get this graph as it is the only possible graph

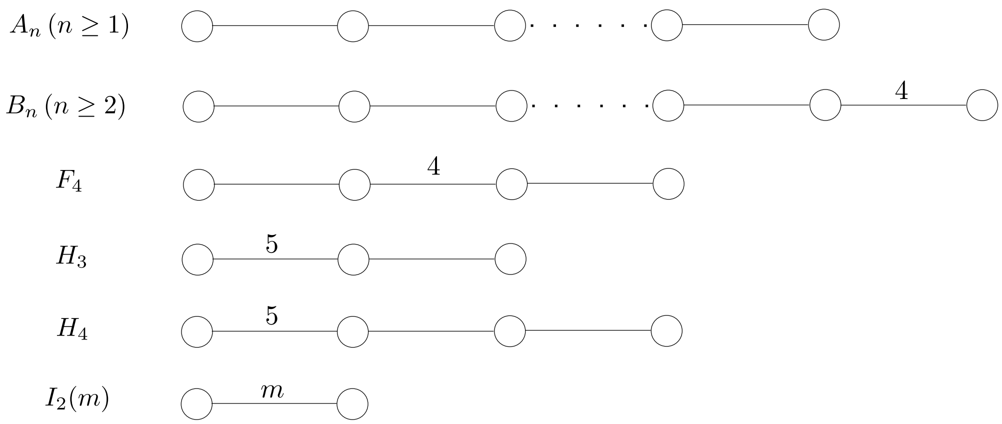
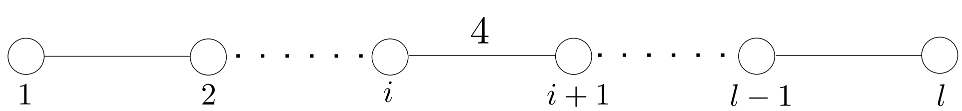


Figure 3.9

of this form, a chain with all edges of order 3.

$B_n\,(n \geq 2)$. We get the B_n as this is a chain with one edge of order 4, where that edge is one of the end edges.

F_4. Assume that $m = 4$ and that the edge with order 4 is not one of the end edges, (in that case we simply have B_n as above). So we have something of the form:



Where $i \geq 2$ and $l - i \geq 2$ since we assume that the edge of order 4 was not at the end. Let $j = l - i$ for simplicity of notation and consider two vectors, v and w such that $v = e_1 + 2e_2 + \cdots + ie_i$ and $w = e_l + 2e_{l-1} + \cdots + je_{i+1}$ then via Lemma 3.4.1,

$$\|v\|^2 = \frac{1}{2}i(i+1) \quad \text{and} \quad \|w\|^2 = \frac{1}{2}j(j+1). \tag{3.5}$$

Also since the edge $i(i+1)$ has order 4 we get:

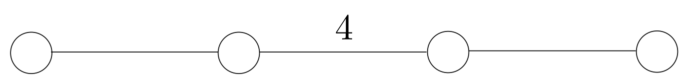
$$(v,w) = -ij(e_i, e_{i+1}) = -ij\cos\left(\frac{\pi}{4}\right) = -\frac{ij}{\sqrt{2}}. \tag{3.6}$$

Using the square of the inner product formula ($(v,w)^2 = \|v\|^2\|w\|^2\cos^2(\theta)$ where θ is the angle between v and w) and the fact that $\cos^2(\theta) < 1$ in this case, we get the relationship $(v,w)^2 < \|v\|^2\|w\|^2$. So using the (3.5) and (3.6) we get,

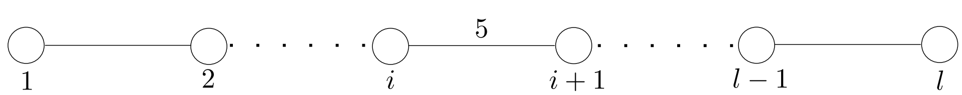
$$\begin{aligned} \left(-\frac{ij}{\sqrt{2}}\right)^2 &< \left(\frac{1}{2}i(i+1)\right)\left(\frac{1}{2}j(j+1)\right), \\ \frac{i^2j^2}{2} &< \frac{i(i+1)j(j+1)}{4}, \\ 2ij &< (i+1)(j+1), \end{aligned}$$

and since $i,j \geq 2$ we must have $i = j = 2$ resulting in the graph of group type F_4.

F_4



$\underline{H_3/H_4}$. Now assume that $m = 5$, we follow a similar argument as previously, the main difference being we do not assume that the edge of order 5 is not an end edge. This gives us a general graph of the form:



with no restrictions on i and j. Again let $j = l - i$ and two vectors v and w such that $v = e_1 + 2e_2 + \cdots + ie_i$ and $w = e_l + 2e_{l-i} + \cdots + je_{i+1}$ then

$$\|v\|^2 = \frac{1}{2}i(i+1) \quad \text{and} \quad \|w\|^2 = \frac{1}{2}j(j+1). \tag{3.7}$$

Also since the edge $i(i+1)$ has order 5 we get:

$$(v, w) = -ij(e_i, e_{i+1}) = -ij \cos\left(\frac{\pi}{5}\right) = -ij\left(\frac{1+\sqrt{5}}{4}\right). \tag{3.8}$$

Again the square of the inner product formula along with (3.7) and (3.8) gives:

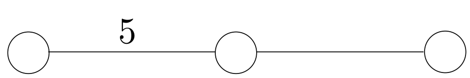
$$\begin{aligned} \left[-ij\left(\frac{1+\sqrt{5}}{4}\right)\right]^2 &< \frac{i(i+1)j(j+1)}{4}, \\ i^2j^2\left(\frac{3+\sqrt{5}}{8}\right) &< \frac{i(i+1)j(j+1)}{4}, \\ ij\left(\frac{3+\sqrt{5}}{2}\right) &< (i+1)(j+1), \end{aligned}$$

and since $3+\sqrt{5} > 5$ we get that,

$$ij\left(\frac{5}{2}\right) < (i+1)(j+1).$$

This forces either $i = 1$ or $j = 1$ so we may assume that $i = 1$ thus $j\left(\frac{5}{2}\right) < 2(j+1)$ which gives $j = 2$ or $j = 3$ leading to the graphs we wanted, namely:

H_3



H_4

□

3.5 Γ has a unique branch point and all edges of order 3

Again in this section we let the vertices of Γ be $\{1, 2, \ldots, l\}$ and focus on graphs of the form given in Figure 3.10.

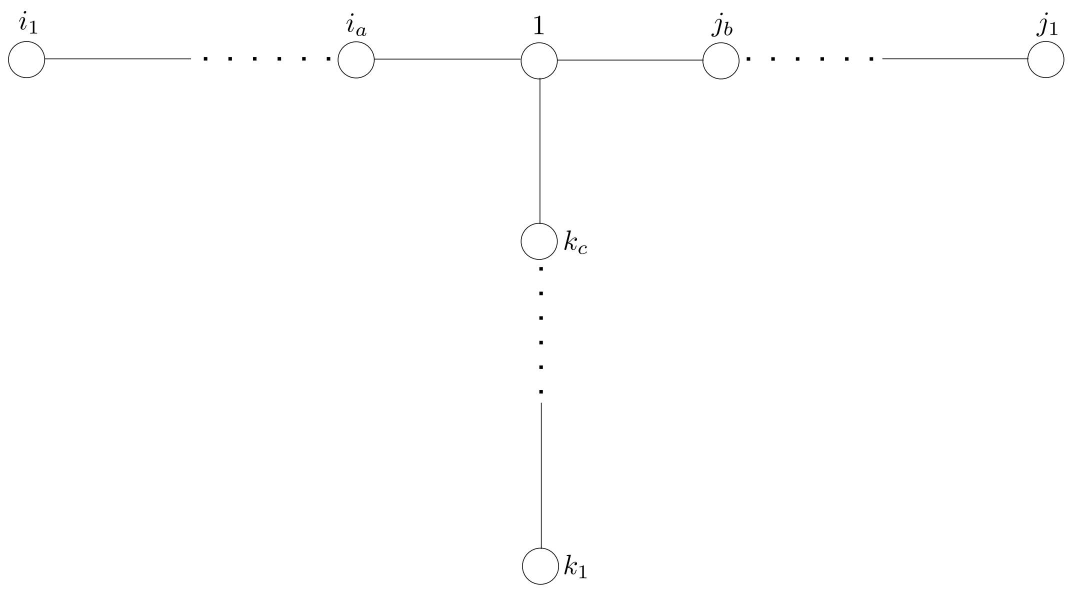


Figure 3.10: Unique branch point with all edges of order 3

Without loss of generality we may assume that $a \geq b \geq c$. We will easily be able to classify the last few positive definite Coxeter graphs after this powerful result.

Lemma 3.5.1. *For a, b and c defined in Figure 3.10,*

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} > 1.$$

Proof. Consider three vectors $u, v, w \in V$ such that,

$$\begin{aligned} u &= e_{i_a} + 2e_{i_{a-1}} + \cdots + ae_{i_a}, \\ v &= e_{j_b} + 2e_{j_{b-1}} + \cdots + be_{j_b}, \\ w &= e_{k_c} + 2e_{k_{c-1}} + \cdots + ce_{k_c}. \end{aligned}$$

Thus each vector u, v and w come from distinct branches of the graph in Figure 3.10, i.e. $(u, v) = (u, w) = (v, w) = 0$ so

$$\left\{ \frac{u}{\|u\|}, \frac{v}{\|v\|}, \frac{w}{\|w\|} \right\}, \tag{3.9}$$

are orthonormal. Also, by Lemma 3.4.1 we get,

$$\begin{aligned} \|u\|^2 &= \frac{1}{2}a(a+1), \\ \|v\|^2 &= \frac{1}{2}b(b+1), \\ \|w\|^2 &= \frac{1}{2}c(c+1). \end{aligned} \tag{3.10}$$

Now let U be the 3 dimensional vector space which is spanned by the vectors in (3.9) and e_1 be the vector corresponding to the branch point vertex in Figure 3.10. As in the proof of Lemma 3.3.2 we let d be the distance from e_1 to U and $\tilde{e_1}$ be the projection of e_1 on U. So via Pythagoras theorem we get $\|\tilde{e_1}\|^2 + d^2 = \|e_1\|^2 = 1$ and since we have $d^2 > 0$ we get,

$$1 - \|\tilde{e_1}\|^2 > 0. \tag{3.11}$$

Now via (3.9) we get an alternative form of $\tilde{e_1}$ which we will combine with the previous to get the result, namely,

$$\tilde{e_1} = \left(e_1, \frac{u}{\|u\|}\right) \frac{u}{\|u\|} + \left(e_1, \frac{v}{\|v\|}\right) \frac{v}{\|v\|} + \left(e_1, \frac{w}{\|w\|}\right) \frac{w}{\|w\|},$$

i.e.

$$\|\tilde{e_1}\|^2 = \frac{(e_1,u)^2}{\|u\|^2} + \frac{(e_1,v)^2}{\|v\|^2} + \frac{(e_1,w)^2}{\|w\|^2}. \tag{3.12}$$

Finally, since e_1 is orthogonal to all vectors except e_{i_a}, e_{j_b} and e_{k_c} we have,

$$(e,u) = -\frac{1}{2}a, \quad (e,v) = -\frac{1}{2}b, \quad (e,w) = -\frac{1}{2}c. \tag{3.13}$$

Now substituting (3.10) and (3.13) into (3.12) we get,

$$\begin{aligned}
\|\tilde{e_1}\|^2 &= \left(-\frac{1}{2}a\right)^2 \cdot \frac{2}{a(a+1)} + \left(-\frac{1}{2}b\right)^2 \cdot \frac{2}{b(b+1)} + \left(-\frac{1}{2}c\right)^2 \cdot \frac{2}{c(c+1)}, \\
&= \frac{2a^2}{4a(a+1)} + \frac{2b^2}{4b(b+1)} + \frac{2c^2}{4c(c+1)}, \\
&= \frac{a}{2(a+1)} + \frac{b}{2(b+1)} + \frac{c}{2(c+1)}.
\end{aligned}$$

Substituting this into (3.11) we get,

$$\begin{aligned}
1 - \left(\frac{a}{2(a+1)} + \frac{b}{2(b+1)} + \frac{c}{2(c+1)}\right) &> 0, \\
1 &> \left(\frac{a}{2(a+1)} + \frac{b}{2(b+1)} + \frac{c}{2(c+1)}\right), \\
2 &> \left(\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1}\right), \\
2 + \left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right) &> \left(\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1}\right) + \\
&\qquad \left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right), \\
2 + \left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right) &> 3, \\
\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} &> 1.
\end{aligned}$$

□

Proposition 3.5.2. *The only possible positive definite graphs with a unique branch point and all edges of order 3 are ones from Figure 3.11 (with corresponding group type next to each).*

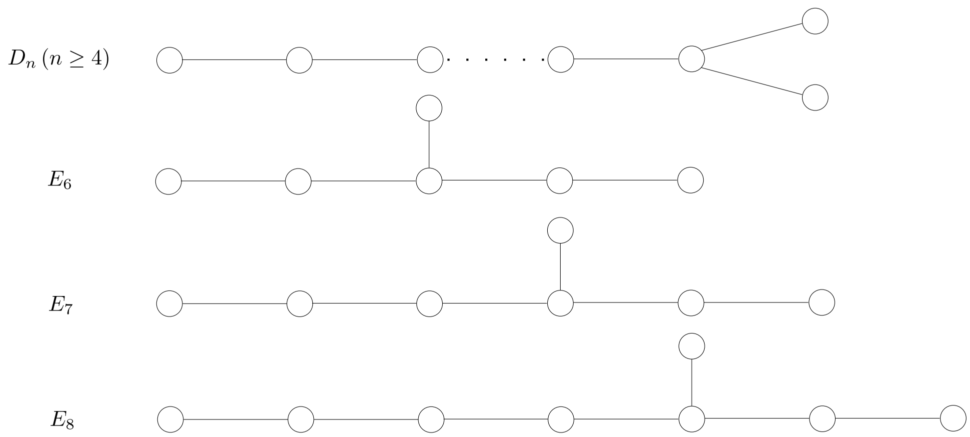


Figure 3.11

Proof. Given $a \geq b \geq c$ from Lemma 3.5.1 we get that $\frac{3}{c+1} > 1$ and thus $c = 1$, so the branch has "length" 1. We now substitute $c = 1$ in to Lemma 3.5.1 to get,

$$\frac{1}{a+1} + \frac{1}{b+1} > \frac{1}{2},$$

and since $a \geq b$ we have $\frac{2}{b+1} > \frac{1}{2}$ i.e. $b = 1$ or $b = 2$.

If $b = 1$ then we substitute this into Lemma 3.5.1, giving $\frac{1}{a+1} > 0$ so a can be anything. Thus we get the graph corresponding to D_n in Figure 3.11.

If $b = 2$ then we get

$$\begin{aligned} \frac{1}{a+1} + \frac{1}{3} &> \frac{1}{2}, \\ a &< 5. \end{aligned}$$

But $a \geq b = 2$, thus $2 \leq a \leq 4$ and so we have $a = 2$, $a = 3$ or $a = 4$ which gives rise to E_6, E_7, E_8 in Figure 3.11. □

3.6 Classification of Real Reflection Groups

So far the results from Chapter 3 have proved the following theorem.

Theorem 3.6.1. *If (W, S) is a finite irreducible Coxeter system, then its Coxeter graph is one of the graphs in Figure 3.12.*

All that is left to do in the classification is to show that each of the graphs in Figure 3.12 give a positive definite form. To check this we need to compute the principal minors of their corresponding matrix.

Definition 3.6.2. The *minor* of an $n \times n$ matrix A is the determinant of some submatrix of A. The *principal minor* is a minor whose submatrix is formed by removing the last k rows and columns of the matrix where $0 \leq k < n$.

Example 3.6.3. The matrix we found in Example 3.2.3,

$$A = \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 1 & -\cos\left(\frac{\pi}{7}\right) \\ -\frac{1}{2} & -\cos\left(\frac{\pi}{7}\right) & 1 \end{pmatrix},$$

has submatrix,

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix},$$

and this submatrix has determinant $\frac{3}{4}$, so $\frac{3}{4}$ is a minor of the matrix A. However it is not a principal minor since the submatrix was not formed by

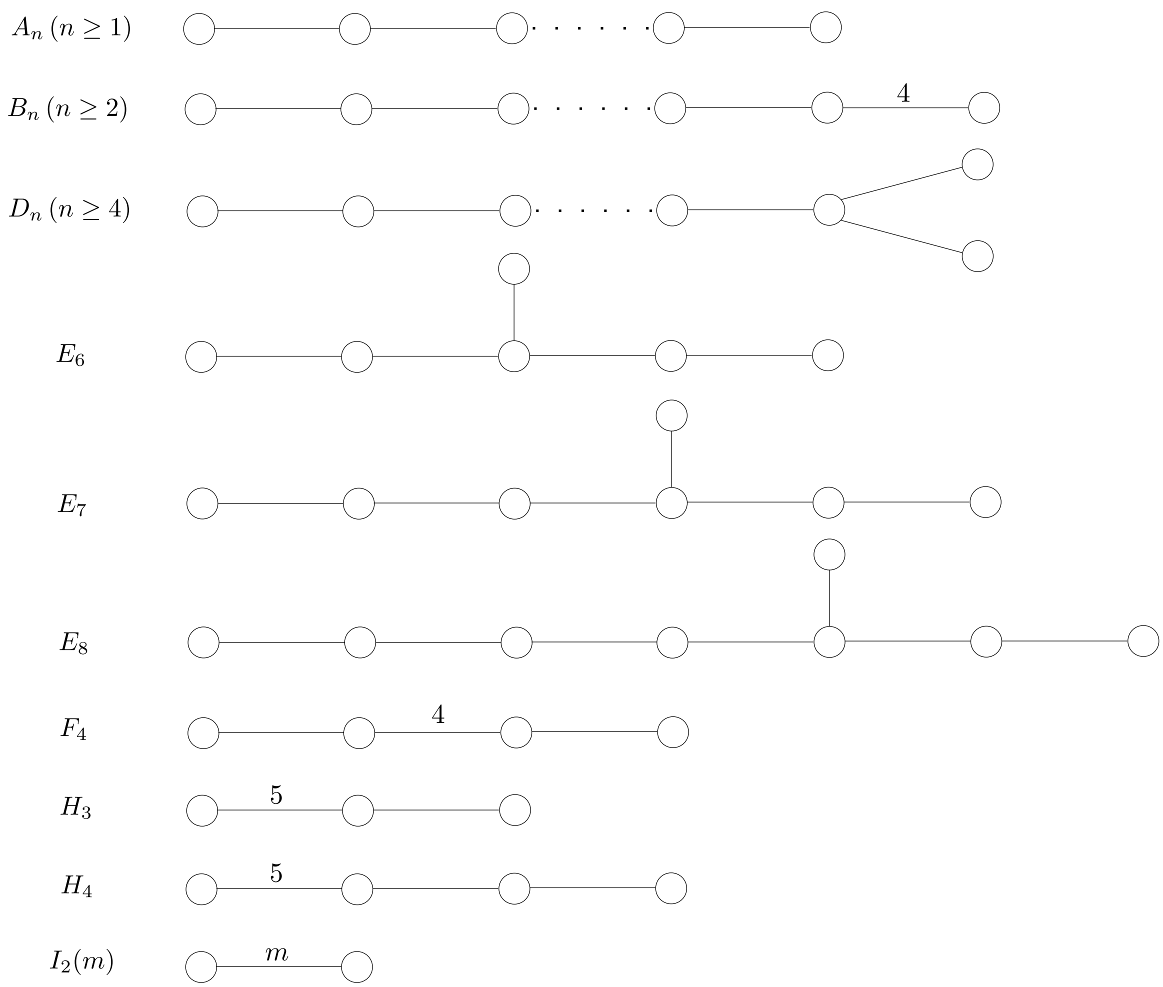


Figure 3.12: Possible Coxeter graphs of finite reflection groups.

removing the last 1 or 2 rows and columns of the matrix. Such submatrices would be,

$$\begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \end{pmatrix},$$

where the first is formed by removing the last row and column of A, and the second formed by removing the last 2 rows and columns of A. These have determinant $\frac{1}{2}$ and 1 respectively i.e. A has principal minors $\frac{1}{2}$ and 1. Since all the principal minors of A are positive we will see in the following lemma that this means the matrix A is positive definite.

Lemma 3.6.4 (Sylvester's Criterion)**.** *A real, symmetric matrix is positive definite if and only if all its principal minors are positive.*

See [24, pg. 328] or [22] for a proof.

Via induction on the number of vertices we can compute $\det A$ (actually we compute $\det 2A$ for simplicity) where A is the corresponding matrix to Γ as each minor is the determinant of a matrix corresponding to a graph from Figure 3.12. It is then easy to check that each of the above graphs do indeed give a positive definite form, specifically, the determinants of the corresponding matrices of each Γ are:

Group Type	**Determinant**
$A_n(n \geq 1)$	$n+1$
$B_n(n \geq 2)$	2
$D_n(n \geq 4)$	4
E_6	3
E_7	2
E_8	1
F_4	1
H_3	$3-\sqrt{5}$
H_4	$\frac{7-3\sqrt{5}}{2}$
$I_2(m)$	$4\sin^2\left(\frac{\pi}{m}\right)$

Thus we have completed the classification of real reflection groups.

Chapter 4

Complex Reflection Groups

> Though analogy is often misleading, it is the least misleading thing we have.
>
> Samuel Butler [9, pg. 59]

The reflections and finite (real) reflection groups we studied in Chapters 2 and 3, over Euclidean vector spaces, can be extended to more general vector spaces. Here we consider the case when the vector space is the complex numbers, i.e. $V = \mathbb{C}$, giving rise to complex reflection groups. Ideally we would like to proceed analogously to Chapters 2 and 3, with some definitions, theorems and then follow this by a classification of all such groups. Although we can classify all such groups, currently, there is no universally accepted theory of root systems, and the subsequent material, for complex reflection groups.

4.1 Extended Definition of a Real Reflection

First we need to define a reflection over the complex numbers.

Definition 4.1.1. Let V be a finite dimensional vector space over $\mathbb{C}$. Then a *reflection on* V is a diagonalizable linear isomorphism $s : V \longrightarrow V$, of finite order, which is not the identity, but fixes pointwise a hyperplane $H \subseteq V$.

We call H the reflecting hyperplane, as in the case for real reflections, all but one eigenvalue of a reflection is equal to 1. However, in the real case, this exceptional value is -1, but for a finite order reflection $s : V \longrightarrow V$, with order n, the exceptional value is an n-th root of unity, ξ, and note that $\det(s) = \xi$.

Definition 4.1.2. For a finite dimensional vector space V, over $\mathbb{C}$, we call the group $G \leq \mathrm{GL}(V)$ a *reflection group* if G is generated by reflections.

Although we do not use it in this chapter, in a similar way to Theorem 2.1.2, we can define a formula for calculating a complex reflection, and we include it for completeness. Recall that for the real numbers $\mathbb{R}$ we had:

$$s_\alpha x = x - \frac{2(x,\alpha)\alpha}{(\alpha,\alpha)}.$$

We just need to extend our idea from an inner product to a *Hermitian form.*

Definition 4.1.3. A *Hermitian form* on a vector space V over $\mathbb{C}$ is a function $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{C}$ such that for all $u, v, w \in V$ and $a, b \in \mathbb{R}$:

1. $(au + bv, w) = a(u, w) + b(v, w)$,
2. $(u, v) = \overline{(v, u)}$,

where $\overline{(v,u)}$ means the complex conjugate of (v,u).
Such a form is *positive definite* if $(x,x) > 0$ for all $0 \neq x \in V$.

For a finite group $G \leq \mathrm{GL}(V)$ we can find a G-invariant positive definite Hermitian form. That is,

$$(g \cdot \alpha, g \cdot \beta) = (\alpha, \beta),$$

for all $\alpha, \beta \in V$ and $g \in G$. If we take any positive definite Hermitian form $(\alpha, \beta)'$ and replace it by,

$$(\alpha, \beta) = \sum_{g \in G} (g \cdot \alpha, g \cdot \beta)',$$

we get another Hermitian form, $(\cdot,\cdot)$, that satisfies $(g(\alpha), g(\beta)) = (\alpha, \beta)$ for all α and β in V. This gives rise to the following remark.

Remark. A reflection $s : V \longrightarrow V$ of order n satisfies:

$$s(x) = x + (\xi - 1)\left(\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha\right),$$

where ξ is an n-th root of unity, α an eigenvector such that $s(\alpha) = \xi\alpha$ and $(\cdot,\cdot)$ is a positive definite G-invariant Hermitian form.

Definition 4.1.4. We call a reflection group $G \leq \mathrm{GL}(V)$ *reducible* if the vector space V is a reducible G-module. That is, it has nontrivial G-submodules. If not, then we call G an *irreducible* complex reflection group.

As in the case for real reflection groups we need only consider irreducible complex reflection groups, since if G is reducible, then G is a direct product of reflection subgroups which are irreducible in smaller dimension. As mentioned before, a systematic approach to roots and root systems for complex reflection groups has not yet been developed. However, in this chapter we do use the term root, and here we give a few definitions to this end.

Definition 4.1.5. A *(unitary) root of a reflection*, in V, is an eigenvector, of length 1, corresponding to the unique nontrivial eigenvalue of the reflection. Notionally, for a reflection $s \in G$ with root α we denote this as α_s.
We similarly define a *(unitary) root of* a group G to be a (unitary) root of a reflection in G.

Definition 4.1.6. As we will use later, we reserve Ω to be the set of all roots from a reflection group G, i.e.

$$\Omega = \{\alpha_s \,|\, \alpha \text{ a (unitary) root corresponding to the reflection } s \in G\}.$$

Before stating one of the main theorems proved by Shephard and Todd (and later by Chevalley), and the most famous result in this project, we recall what is meant by the symmetric algebra. First, pick a basis $\{e_1, \dots, e_n\}$ of the vector space V and let $X_1, \dots, X_n$ be the basis of the dual space, V^*, the set of all linear functionals. We then define the symmetric algebra, S, as $S = \mathbb{C}[V] = \mathbb{C}[X_1, \dots, X_n]$ to be polynomial functions on V, and define an action of G on $\mathbb{C}[V]$ via $(gf)(v) = f(g^{-1}v)$. We denote the subalgebra of G-invariant polynomials, f_i for $1 \leq i \leq n$, by $S^G := \mathbb{C}[f_1, \dots, f_n]$.

Theorem 4.1.7 (Shephard-Todd (or Chevalley-Shephard-Todd))**.** *For a finite group G of linear transformations of V the following are equivalent:*

1. *G is a reflection group in V.*

2. *There are n algebraically independent homogeneous polynomials, $f_1, \dots, f_n \in S^G$ with*
$$|G| = \deg(f_1) \cdot \deg(f_2) \cdot \dots \cdot \deg(f_n).$$

This was proved on a case by case basis by Shephard and Todd in [39], however, we refer the reader to the proof given shortly afterwards by Chevalley [14].

The finite irreducible complex reflection groups were first classified by Geoffrey Shephard and John Todd in 1954 [39] and were later reclassified, in a more modern style, by Arjeh Cohen [15]. The finite irreducible complex reflection groups are divided into 37 cases, there are three infinite families and 34 exceptional groups. The three infinite families are the symmetric groups, the cyclic groups $\mathbb{Z}/m\mathbb{Z}$ and the imprimitive reflection groups, $G(m, p, n)$. The list of irreducible complex reflection groups can be found in Table 4.1 where the groups are ordered in terms of their Shephard-Todd number (given in their classification) and where $\mathrm{Es}_p(n)$ is the extra special group of order p^n. The table also contains what order of reflections, and how many of each, are in the reflection groups and what the rank (which we define below) of each reflection group is.

We define the *rank* of a complex reflection group to be the dimension of the complex vector space on which the group acts. Complex reflection groups of rank 1 must be the cyclic groups, $\mathbb{Z}/m\mathbb{Z}$. If the rank is ≥ 2 then we consider three cases, either it is an imprimitive reflection group, a primitive reflection group of rank 2, or of rank ≥ 3. Here we only give the complete proof of the classification of the imprimitive reflections group. However we will later touch upon the cases for primitive reflection groups.

4.2 Imprimitive Reflection Groups

To classify the imprimitive reflection groups we follow a similar argument as given by [15].

Table 4.1: All Possible Complex reflection groups

S-T Number	Rank	Name	Reflections
1	$n-1$	$G(1,1,n) = \mathrm{Sym}(n)$	$2^{\frac{n(n-1)}{2}}$
2	n	$G(m,p,n)$ for $m>1, p>1$ $(m,p,n \neq 2)$ and $p \mid m$	$2^{\frac{mn(n-1)}{2}}$, $d^{n\phi(d)}$ for $d \mid \frac{m}{p}$ and $d>1$
3	1	$G(m,1,1) = \mathbb{Z}_m$	$d^{\phi(d)}$ for $d \mid m$, $d>1$
4	2	$\mathbb{Z}_2 \cdot \mathrm{Alt}(4)$	3^8
5	2	$\mathbb{Z}_6 \cdot \mathrm{Alt}(4)$	3^{16}
6	2	$\mathbb{Z}_4 \cdot \mathrm{Alt}(4)$	$2^6, 3^8$
7	2	$\mathbb{Z}_{12} \cdot \mathrm{Alt}(4)$	$2^6, 3^{16}$
8	2	$\mathbb{Z}_4 \cdot \mathrm{Sym}(4)$	$2^6, 4^{12}$
9	2	$\mathbb{Z}_8 \cdot \mathrm{Sym}(4)$	$2^{18}, 4^{12}$
10	2	$\mathbb{Z}_{12} \cdot \mathrm{Sym}(4)$	$2^6, 3^{16}, 4^{12}$
11	2	$\mathbb{Z}_{24} \cdot \mathrm{Sym}(4)$	$2^{18}, 3^{16}, 4^{12}$
12	2	$\mathbb{Z}_2 \cdot \mathrm{Sym}(4) = \mathrm{GL}_2(\mathbb{F}_3)$	2^{12}
13	2	$\mathbb{Z}_4 \cdot \mathrm{Sym}(4)$	2^{18}
14	2	$\mathbb{Z}_6 \cdot \mathrm{Sym}(4)$	$2^{12}, 3^{16}$
15	2	$\mathbb{Z}_{12} \cdot \mathrm{Sym}(4)$	$2^{18}, 3^{16}$
16	2	$\mathbb{Z}_{10} \cdot \mathrm{Alt}(5)$	5^{48}
17	2	$\mathbb{Z}_{20} \cdot \mathrm{Alt}(5)$	$2^{30}, 5^{48}$
18	2	$\mathbb{Z}_{30} \cdot \mathrm{Alt}(5)$	$3^{40}, 5^{48}$
19	2	$\mathbb{Z}_{60} \cdot \mathrm{Alt}(5)$	$2^{30}, 3^{40}, 5^{48}$
20	2	$\mathbb{Z}_6 \cdot \mathrm{Alt}(5)$	3^{40}
21	2	$\mathbb{Z}_{12} \cdot \mathrm{Alt}(5)$	$2^{30}, 3^{40}$
22	2	$\mathbb{Z}_4 \cdot \mathrm{Alt}(5)$	2^{30}
23	3	$\mathbb{Z}_2 \cdot \mathrm{PSL}_2(5)$	2^{15}
24	3	$\mathbb{Z}_2 \cdot \mathrm{PSL}_2(7)$	2^{21}
25	3	$\mathrm{Es}_3(3) \cdot \mathrm{SL}_2(3)$	3^{24}
26	3	$\mathbb{Z}_2 \times \mathrm{Es}_3(3) \cdot \mathrm{SL}_2(3)$	$2^9, 3^{24}$
27	3	$\mathbb{Z}_2 \times (\mathbb{Z}_3 \mathrm{Alt}(6))$	2^{45}
28	4	$(\mathrm{SL}_2(3) \times \mathrm{SL}_2(3)) \cdot (\mathbb{Z}_2 \times \mathbb{Z}_2)$	2^{24}
29	4	$(\mathbb{Z}_4 \times \mathrm{Es}_2(5)) \cdot \mathrm{Sym}(5)$	2^{40}
30	4	$(\mathrm{SL}_2(5) \times \mathrm{SL}_2(5)) \cdot \mathbb{Z}_2$	2^{60}
31	4	$(\mathbb{Z}_4 \times \mathrm{Es}_2(5)) \cdot \mathrm{Sp}_4(2)$	2^{60}
32	4	$\mathbb{Z}_3 \times \mathrm{Sp}_4(3)$	3^{80}
33	5	$\mathbb{Z}_2 \times \mathrm{PSU}_4(2)$	2^{45}
34	6	$\mathbb{Z}_3 \cdot \mathrm{SO}_6(3)$	2^{126}
35	6	$\mathrm{PSU}_4(2) \cdot \mathbb{Z}_2$	2^{36}
36	7	$\mathbb{Z}_2 \times \mathrm{Sp}_6(2)$	2^{63}
37	8	$\mathbb{Z}_2 \cdot \mathrm{O}_8(2)$	2^{120}

Definition 4.2.1. For any $m, p, n \geq 1$, and $p|m$ (p a factor of m) we define $G(m, p, n)$ to be the group of monomial $n \times n$ matrices (those with only one nonzero entry in each row and column) where the nonzero entries are mth roots of unity, and the product of these entries is a $\frac{m}{p}$th root of unity.

Remark. $G(m, p, n)$ can also be defined via the semi-direct product,

$$G(m, p, n) := A(m, p, n) \rtimes \Pi_n,$$

where $A(m, p, n) :=$ the group of all diagonal $n \times n$ matrices (nonzero entries only appear on leading diagonal) such that each of these entries are mth roots of unity (i.e. $\xi_i^m = 1$ for each $i = 1, \ldots, n$) with the further condition that $(\det A)^{\frac{m}{p}} = 1$, where $A \in A(m, p, n)$. $\Pi_n :=$ the group of $n \times n$ permutation matrices, that is $n \times n$ monomial binary matrices.

Note that since the order of the symmetric group on n letters is $n!$ we have that $|\Pi_n| = n!$. Also $|A(m, p, n)| = \frac{m^n}{p}$ since there are m^n possibilities for $n \times n$ diagonal matrices with mth roots of unity on the diagonal but since we require $(\det A)^{\frac{m}{p}} = 1$ and the determinant of a diagonal matrix is nothing more than the product of the diagonal entries we have counted p times too many elements, thus,

$$|G(m, p, n)| = \frac{m^n n!}{p}.$$

The only conjugates which appear in this set of groups is $G(2, 1, 2)$ conjugate to $G(4, 4, 2)$. We discuss this fact further after Lemma 4.2.12. The only reducible case is for $(m, p, n) = (2, 2, 2)$.

Examples 4.2.2. The $G(m, p, n)$ give rise to the real reflection groups, for example,

$$\begin{aligned} G(m,m,2) = \mathbb{Z}/m\mathbb{Z} \rtimes \Pi_2 &= D_{2m} = W(I_2(m)), && \text{(dihedral group)} \\ G(1,1,n) &= W(A_{n-1})\,(n \geq 2), && \text{(symmetric group)} \\ G(2,1,n) = (\mathbb{Z}/m\mathbb{Z})^n \rtimes \Pi_n &= W(B_n), \\ G(2,2,n) = (\mathbb{Z}/m\mathbb{Z})^{n-1} \rtimes \Pi_n &= W(D_n). \end{aligned}$$

Definition 4.2.3. A group $G \leq \mathrm{GL}(V)$ is called *imprimitive* if there exists a decomposition of the vector space $V = \bigoplus_{i=1}^k V_i = V_1 \oplus \cdots \oplus V_k$ for $k > 2$ where the subspaces $(V_i)_{1\leq i\leq k}$ are nontrivial proper linear subspaces of V and are permuted transitively by G (as in Definition 1.2.7 there exists a transitive group action of G on $\{V_i \,|\, 1 \leq i \leq k\}$). We call the $(V_i)_{1\leq i\leq k}$ a *system of imprimitivity* for G.

We can similarly define the decomposition $V = \bigoplus_{i=1}^k V_i$ to be imprimitive if for all $g \in G$ and for all V_i we get $gV_i = V_j$ for some j, and for all V_i and V_j there exists a $h \in G$ such that $hV_i = V_j$.

Proposition 4.2.4. *If G is an irreducible imprimitive reflection group in V, with dimension n, and $(V_i)_{1\leq i\leq k}$ is a system of imprimitivity for G, then:*

1. *For $1 \leq i \leq k$ we have $\dim V_i = 1$*
2. *For an arbitrary reflection $s \in G$, either:*
   (a) *$sV_i = V_i$ for $1 \leq i \leq n$, or,*
   (b) *there exists $i \neq j$, for $1 \leq i, j \leq n$, such that any root of s is contained in $V_i + V_j$, where $sV_i = V_j$ and $sV_k = V_k$ for all $k \neq i, j$ and s is an involution, that is, of order 2.*

Proof. 1. For a contradiction fix i such that $\dim V_i > 1$. Then since G is irreducible there exists a j such that $j \neq i$ and a reflection $s \in G$ such that $sV_i = V_j$. Thus $\dim(V_j \cap V_i) > 0$, which contradicts the fact that $V_i \cap V_j = \{0\}$, so $\dim V_i = 1$.

2. Let $s \in G$ be a reflection with root α_s, we first assume that $(\alpha_s, V_i) \neq 0$ (where $(\cdot,\cdot)$ is the inner product over $\mathbb{C}$) for $i = 1$ or $i = 2$. Let ξ be a nontrivial eigenvalue of s such that $(\alpha_s, V_1) \neq 0$ and $\alpha_s \notin V_1$. We know $sV_1 = V_2$ since the group is imprimitive. Now consider a nonzero $x_i \in V_i$ for $i = 1$ or $i = 2$ where $sx_1 = x_2$. Then there exists a $j \in \{1, \dots, n\}$ with $s^2x_1 \in V_j$ and so $s^2x_1 \in (\mathbb{C}\alpha_s + \mathbb{C}x_1) \cap V_j$. But $(\mathbb{C}\alpha_s + \mathbb{C}x_1) \cap V_j = (V_1 + V_2) \cap V_j$, which implies that $j = 1$ or $j = 2$. But since $\alpha_s \notin V_1$ we have that $s^2x_1 = x_1$ and $\xi^2 = 1$ which means that $\xi = -1$ and s is of order 2. Then, as α_s is a scalar multiple of $(x_1 - x_2)$, we get that $\alpha_s \in V_1 + V_2$.

   We now consider $(\alpha_s, V_i) \neq 0$ for $i \geq 3$, thus there exists $j \geq 3$ such that $sV_i = V_j$. Which forces $\alpha_s \in (V_i + V_j) \cap (V_1 + V_2) = \{0\}$, a contradiction. Thus $(\alpha_s, V_i) = 0$ for $i \geq 3$ and so $sV_k = V_k$ for $k \geq 3$. □

We will now show that the $G(m, p, n)$ are irreducible in most cases.

Lemma 4.2.5. *$G(m, p, n)$ is irreducible if and only if $m > 1$ and $(m, p, n) \neq (2, 2, 2)$.*

Proof. "⇒" Assume that $G(m, p, n)$ is irreducible, in which case $G(m, p, n)$ fixes a nontrivial proper subspace W of V. Since W is a Π_n-invariant subspace of V, we get $W = \mathbb{C}(e_1 + e_2 + \cdots + e_n)$, up to switching between W and $W^\perp := \{v \in V \mid (w, v) = 0\ \forall w \in W\}$. Now as $A(m, p, n)$ stabilizes $\mathbb{C}(e_1 + \cdots + e_n)$ all the diagonal coefficients of any of the elements in $A(m, p, n)$ must be equal, this implies that we must have $(m, p, n) = (1, 1, n)$ or $(m, p, n) = (2, 2, 2)$.

"$\Leftarrow$" We prove the contrapositive, if $m = 1$ then we must have $G(1, 1, n)$ which is clearly reducible since this is symmetric group on $n - 1$ letters. Otherwise we must have $G(2, 2, 2) \cong V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, the Klein four-group, which is reducible in V. $\square$

From here onwards we use the set of unitary complex numbers in a number of our proofs, so we first formally define them.

Definition 4.2.6. The set of unitary complex numbers, $\mathbb{U}$, are those which when multiplied with their complex conjugate give 1, i.e.

$$z\overline{z} = 1,$$

for all $z \in \mathbb{U}$.

The $G(m, p, n)$ are imprimitive reflection groups in $\mathbb{C}^n$, and the next theorem proves exactly this. The system of imprimitivity for the $G(m, p, n)$ is $(\mathbb{C}e_i)_{1 \leq i \leq n}$, where the e_i make up the standard basis of $\mathbb{C}^n$.

Theorem 4.2.7. *Consider an irreducible imprimitive reflection group G on a vector space V, where $\dim V \geq 2$. Then G is conjugate to $G(m, p, n)$ for $m, p \in \mathbb{N}$, where $p|m$.*

Proof. Let G be an irreducible imprimitive reflection group on a vector space V. Then we can find an orthonormal basis of V, say $\{\epsilon_1, \ldots, \epsilon_n\}$ such that $V_i = \mathbb{C}_i\epsilon_i$, for $1 \leq i \leq n$, form a system of imprimitivity for G. Then for $j > 1$ there is a corresponding reflection $s_j \in G$ such that $s_j\epsilon_1 = \epsilon_j$ from Proposition 4.2.4. Without changing the conjugacy class of G we see that in fact $\{\epsilon_1, \ldots, \epsilon_n\}$ is the standard basis. So we relabel it $\{e_1, \ldots, e_n\}$, in line with our notation. It again follows from Proposition 4.2.4 that Π_n (i.e. the group of $n \times n$ permutation matrices) is a subgroup of G.

Now consider the cyclic group generated by the reflections which leave a hyperplane (without loss say H_{e_1}) of V fixed pointwise, and let the order of such a group be q. Thus $A(q,1,n)$ (defined as in the remark after Definition 4.2.1) is a subgroup of G.

Again, via Proposition 4.2.4, the only reflections outside of $A(q,1,n) \rtimes \Pi_n$ are $s' \in G$ with $s'e_i = \varphi e_j$, for $\varphi \in \mathbb{U}\backslash\{1\}$ with $i \neq j$ and $s'e_k = e_k$ for all $k \neq i,j$. Then we can take $i = 1$ and $j = 2$ up to conjugacy by an element of Π_n. Let $s = s_2 \in G$ be a reflection with $se_1 = e_2$, then $(ss')e_1 = \varphi e_1$ and $(ss')e_2 = \varphi^{-1}e_2$, thus φ is a root of unity. Finally, for a reflection $t \in G$ such that $tV_1 = V_2$, we define m to be the maximum order of the elements $st \in G$. So we see that $A(m,m,n) \leq G$.

Now let $p = \frac{m}{q}$, then we have that $A(m,p,n) = \langle A(q,1,n), A(m,m,n)\rangle$ so $G(m,p,n) = A(m,p,n) \rtimes \Pi_n \leq G$. But all reflections of G must be contained in this subgroup, so the subgroup must in fact be equal to the group G, i.e. $G = G(m,p,n)$. □

Definition 4.2.8. Recall that $\mathbb{U}$ is the set of unitary complex numbers and $\Omega = \{\alpha_s \,|\, \alpha$ a (unitary) root corresponding to a reflection $s \in G\}$, let $P = \{\mathbb{U}\alpha_s \,|\, \alpha$ a root of a reflection $s \in G\}$. Since G acts on P there is a map $\tau : P \longrightarrow \Omega$ where, for $L \in P$, we define $\tau(L) = \alpha$ if and only if $\alpha \in L \cap \Omega$. If $\mathcal{O}$ is an orbit of G in P define $f_{\mathcal{O}} \in S$ by $f_{\mathcal{O}} = \prod_{L\in\mathcal{O}} l_{\tau(L)}$.

Definition 4.2.9. We define another map, $\chi_{\mathcal{O}}$, with $\mathcal{O}$ the same as above, by $\chi_{\mathcal{O}} : G \longrightarrow \mathbb{U}$ via

$$\chi_{\mathcal{O}}(g) = \prod_{\mathbb{U}\alpha_{s_i}\in\mathcal{O}} \left(\frac{1}{\det s_i}\right),$$

where $\{s_1, \ldots, s_r\}$ are the reflections of G and $g = s_1 \cdot s_2 \cdots s_r$.

Definition 4.2.10. If V is a $\mathbb{C}G$-module then the function:

$$\begin{array}{rcl}\mathbb{C}[G] & \longrightarrow & \mathbb{C} \\ g & \longmapsto & Tr(\rho(g))\end{array}$$

where $\rho : G \longrightarrow \mathrm{GL}(V)$ is a representation of G on V and Tr is the trace, is called the $\mathbb{C}$ character of V.

In the following proposition we see that characters of G are products of the above $\chi_{\mathcal{O}}$'s.

Proposition 4.2.11. *Any linear character of G is the product of some $\chi_{\mathcal{O}}$, where $\chi_{\mathcal{O}}$ is defined as in Definition 4.2.9.*

Proof. We prove this by induction. For a base case let ϕ be any nontrivial linear character of G, and let $f \in S$ be a nonzero homogeneous polynomial of minimum degree such that $gf = \phi(g)f$ for $g \in G$. Such f exist as S/S^G is isomorphic to the regular module (the group algebra considered as a module and where any irreducible module occurs as a submodule). Now, if $s \in G$ is a reflection such that $\phi(s) \neq 1$ then for any $v \in V$ such that $(v, \alpha_s) = 0$ we must have that

$$f(v) = f(s^{-1}v) = (sf)v = \phi(s)f(v),$$

which implies that $f(v) = 0$. Hence f is divisible by l_{α_s} and $l_{\tau(L)}$ for any L in the G-orbit $\mathcal{O}$ of $\mathbb{U}\alpha_s$, thus f is divisible by $f_{\mathcal{O}}$.

Now define $f_1 := \frac{f}{f_{\mathcal{O}}}$ and $\phi_1 := \frac{\phi}{\chi_{\mathcal{O}}}$. If f_1 is not a constant then it is a nonzero homogeneous polynomial of minimum degree such that $gf_1 = \phi_1(g)f_1$, again for $g \in G$, and has degree strictly lower than the degree of f, so we are done by induction. □

Remark. $G = G(m, p, 2)$ consists of r G-orbits where $r = \frac{hcf(2,m)}{hcf\left(2,\frac{m}{p}\right)}$ each of length $\frac{m \cdot hcf\left(2,\frac{m}{p}\right)}{hcf(2,m)}$.

Lemma 4.2.12. *The degrees of $G(m,p,n)$ are $m, 2m, \ldots, (n-1)m, \frac{m}{p}n$.*

Proof. Let $X_1, X_2, \ldots, X_n$ be polynomial functions corresponding to the standard basis $\{e_1, \ldots, e_n\}$ of V, such that $X_1, \ldots, X_n \in S$ (where $X_i : V \longrightarrow \mathbb{C}$ for $1 \leq i \leq n$) and let $q = \frac{m}{p}$. Then the first $n-1$ elementary symmetric polynomials in $(X_i^m)_{1 \leq i \leq n}$ (those of the form $X_i(e_j) = \delta_{ij}$) and $(X_1 \cdot X_2 \cdots X_n)^q$ form a set of $G(m,p,n)$ invariant homogeneous algebraically independent polynomials and,

$$\begin{aligned} m \cdot 2m \cdot \cdots \cdot (n-1)m \cdot qn &= qm^{n-1}n!, \\ &= \frac{m^n n!}{p}, \\ &= |G(m,p,n)|, \end{aligned}$$

and so by Theorem 4.1.7 the degrees of $G(m,p,n)$ are $m, 2m, \ldots, (n-1)m$ and $qn = \frac{m}{p}n$. □

Corollary 4.2.13. *The order of the centre of the $G(m,p,n)$'s is equal to $\frac{hcf(p,n) \cdot m}{p}$.*

Proof. An element of the centre must be a diagonal matrix, i.e $A := \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$ where A is an $n \times n$ matrix. Then, since we know that the determinant of a diagonal matrix is the product of the diagonal elements, $\det A = \lambda^n$. We also have the extra condition, since $A \in G(m,p,n)$ that $(\det A)^{\frac{m}{p}} = 1$, i.e. (we let $q = \frac{m}{p}$ as before)

$$(\det A)^{\frac{m}{p}} = (\lambda^n)^{\frac{m}{p}} = \lambda^{\frac{nm}{p}} = \lambda^{nq} = 1.$$

The number of solutions to this equation is,

$$\begin{aligned} hcf(nq, m) &= hcf(nq, pq), \\ &= q \cdot hcf(n, p), \\ &= \frac{m \cdot hcf(n, p)}{p}. \end{aligned}$$

Thus $|\mathrm{Z}(G(m, p, n))| = \frac{m \cdot hcf(n,p)}{p}$. □

As was previously mentioned the only conjugates in the $G(m, p, n)$ are $G(2, 1, 2)$ and $G(4, 4, 2)$. To show this consider two groups $G_1 := G(m, p, n)$ and $G_2 := G(m', p', n)$, these groups will be conjugate if and only if there degrees are equal. We fix $n = 2$ and then we see that the degrees of G_1 are m and $\frac{2m}{p}$ while the degrees of G_2 are m' and $\frac{2m'}{p'}$. We cannot have that $m = m'$ and $\frac{2m}{p} = \frac{2m'}{p'}$ otherwise we would just get that $G_1 = G_2$ so we must have $m = \frac{2m'}{p'}$ and $m' = \frac{2m}{p}$. Via rearranging the first equation and substituting the second into it we get that

$$\begin{aligned} p' &= \frac{2m'}{m}, \\ p' &= \frac{2\left(\frac{2m}{p}\right)}{m}, \\ mp' &= \frac{4m}{p}, \\ pp' &= 4. \end{aligned}$$

So we have two cases to consider (without loss of generality for the second case we assume $p = 1$, we could let p' equal 1, but that will result in exactly the same result up to renaming G_2 as G_1, and vice versa).

<u>Case 1:</u> $p = p' = 2$

In this case G_1 and G_2 both just have a single degree (m and m' respectively), forcing $n = 1$ and so $G_1 = G_2$.

Case 2: $p = 1$ and $p' = 4$

So the degrees of G_1 are m and $2m$ while the degrees of G_2 are m' and $\frac{m'}{4}$ and this condition forces us to have $m = 2$ and $m' = 4$.

Thus we have that $G(2, 1, 2)$ and $G(4, 4, 2)$ have the same degrees, are conjugate and are the only complex reflection groups of rank 2 which do. A similar argument can be used for $n > 2$.

Theorem 4.2.14. *Suppose $G = G(m, p, n)$ is irreducible, i.e. $p|m$ and $n \geq 2$, then G has a unique system of imprimitivity if $(m, p, n) \notin \{(2, 1, 2), (4, 4, 2), (3, 3, 3), (2, 2, 4)\}$.*

Proof. Let a basis of the vector space V be $\{e_1, \ldots, e_n\}$. Then $\mathbb{C}e_i$ for $1 \leq i \leq n$ is a system of imprimitivity for $G(m, p, n)$. Again let $P = \{\mathbb{U}\alpha_s \,|\, \alpha$ a root of a reflection $s \in G\}$ and consider an orbit of P that gives another system of imprimitivity. Via the remark after Proposition 4.2.11, with $q = \frac{m}{p}$, we have that:

$$n = 2 = \frac{m \cdot hcf(2, q)}{hcf(2, m)}, \tag{4.1}$$

or

$$n > 2 \;\text{ and }\; mn(n-1) = 2n. \tag{4.2}$$

Now if we have (4.1) we get that (m, p, n) must be either $(2, 1, 2)$ or $(4, 4, 2)$. (4.2) gives us a contradiction in Theorem 4.2.7 with $m > 1$. So none of the $G(m, p, n)$ have a system of imprimitivity coming from roots other than those of the canonical system.

We now work to show that $(m, p, n) \notin \{(2, 2, 4), (3, 3, 3)\}$ if $G(m, p, n)$ has a unique system of imprimitivity (in fact we show $(m, p, n) \notin \{(2, 1, 2), (2, 2, 4), (3, 3, 3)\}$). Assume that $(V_i)_{1 \leq i \leq n}$ form a system of imprimitivity different from the one formed by $L_1, \dots, L_n$ and that it does not correspond to any orbit of P. We define $l_1, \dots, l_n$ to be distinct (not equal up to a constant factor) linear homogeneous polynomials of degree n with respect to $V_1, \dots, V_n$ (as in Proposition 4.2.4).

Let $f = l_1 \cdot l_2 \cdots l_n$, and for a contradiction assume that f is semi-invariant ($f(gx) = c \cdot f(x)$, for a constant $c \in \mathbb{C}$ and for all $g \in G$), but not necessarily invariant. Then in the same was as Proposition 4.2.11 we get that f is the product of an invariant and some $f_{\mathcal{O}}$ where $\mathcal{O}$ is an orbit of P. By definition of f, $\deg f = n$ and since G is irreducible there is an orbit $\mathcal{O}$ in P of length n such that $f = f_{\mathcal{O}}$, a contradiction. Thus f is invariant.

We know that m must divide n since $f \notin \mathbb{C}(X_1 \cdot X_2 \cdots X_n)$ implies there exists $\alpha \in \mathbb{C}$ such that $f - \alpha(X_1 \cdots X_n)$ is a nonzero homogeneous G-invariant polynomial in $(X_i^m)_{1 \leq i \leq n}$ as in Lemma 4.2.12.

Finally we let $l_1 = \gamma_1 X_1 + \gamma_2 X_2 + \cdots + \gamma_n X_n$ where the $\gamma_i \in \mathbb{C}$. Then define r to be the number of γ_i's which are the same, i.e. $r = |\{\gamma_k \,|\, \gamma_k = \gamma_i \text{ for } 1 \leq i \leq n \text{ and } k \neq i\}|$, and r_0 to be the number of nonzero γ_i's, i.e. $r_0 = |\{\gamma_i \,|\, \gamma_i \neq 0 \text{ for } 1 \leq i \leq n\}|$, note that $r_0 \neq 1$. Then since the stabilizer in Π_n of $\mathbb{C}l_1$ has order less than or equal to $r!(n-r)!$ and the size of the Π_n orbit of $\mathbb{C}l_1$ is less than or equal to n, we must have,

$$n \geq \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Thus $r = 1$, $r = n-1$ or $r = n$.

Since $r_0 \neq 1$ we have $r_0 = n$ as if $r_0 = n-1$ then the stabilizer of $\mathbb{C}l_1$ in $G(m,p,n)$ would have order $\leq mq(n-1)!$. Thus $n \geq \frac{m^{n-1}qn!}{mq(n-1)!} = m^{n-2}n$ which implies that $n \leq 2$ and $l_1 \in \mathbb{C}X_1$, but this contradicts that $V_1, V_2, \dots, V_n$ is different from $L_1, \dots, L_n$, so $r_0 \neq n-1$, hence $r_0 = n$. So all of the γ_i's are nonzero. We can conclude that the $\mathbb{C}l_1$ stabilizer in $G(m,p,n)$ is $\leq mn!$ and so

$$n \geq \frac{m^{n-1}qn!}{mn!} = m^{n-2}q = m^{n-2} \cdot \frac{m}{p} = \frac{m^{n-1}}{p}.$$

Combining this with the fact that m is a factor of n (as we worked out earlier in the proof) we have that (m, p, n) must be either $(2, 1, 2)$, $(2, 2, 4)$ or $(3, 3, 3)$.

Thus $G(m, p, n)$ has a unique system of imprimitivity if $(m, p, n) \notin \{(2, 1, 2),$ $(4, 4, 2)$, $(3, 3, 3)$, $(2, 2, 4)\}$. $\square$

This completes the classification of the imprimitive complex reflection groups since we have shown that if $m > 1$ and $(m, p, n) \neq (2, 2, 2)$ for $G(m, p, n)$ then then $G(m, p, n)$ is an irreducible reflection group (this is via Lemma 4.2.5 and Theorem 4.2.7). In particular the imprimitive complex reflection groups contain a unique system of imprimitivity as long as $(m, p, n) \notin \{(2, 1, 2), (4, 4, 2), (3, 3, 3), (2, 2, 4)\}$, this is via Theorem 4.2.14.

4.3 Primitive Complex Reflection Groups

We have seen in the classification of the imprimitive reflection groups that such groups give rise to an infinite family of complex reflection groups. However, as has been previously mentioned, we still have one infinite family and 34 exceptional cases unaccounted for. These are the primitive complex reflection groups, and although we do not give a full proof of the classification of such groups here, we briefly discuss them an refer the reader to [15] for a full proof. The proof is split into two cases, the primitive reflections groups of rank 2 and of rank ≥ 3.

4.3.1 Rank 2

Of the 34 exceptional cases 19 of them are primitive complex reflection groups of rank 2.

We need to find a group G generated by reflections such that $G \leq \mathrm{GL}_2(\mathbb{C})$. Since we can decompose $\mathrm{GL}_2(\mathbb{C})$ into $\mathbb{C} \times \mathrm{SL}_2(\mathbb{C})$ and the conjugacy classes of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are known, this is fairly simple to find. $\mathrm{SL}_2(\mathbb{C})$ contains as its conjugacy classes, $\mathbb{Z}/m\mathbb{Z}$, D_{2m} (the dihedral group) and for $k = 3, 4, 5$ groups with the presentation $\langle x, y \,|\, x^2 = y^3 = (xy)^k = 1\rangle$ (that is the tetrahedral, octahedral and icosahedral groups respectively). Then for subgroups K and H of $\mathrm{SL}_2(\mathbb{C})$ such that

$$H \lhd K \leq \mathrm{SL}_2(\mathbb{C}) \quad \text{and} \quad K/H = \mathbb{Z}/m\mathbb{Z},$$

we see that we can get all subgroups of $\mathrm{GL}_2(\mathbb{C})$ by defining a group G such that $G \leq \mathbb{Z}/km\mathbb{Z} \times K \leq \mathrm{GL}_2(\mathbb{C})$, for each $k \geq 2$ and where $\mathbb{Z}/k\mathbb{Z} \lhd G$ and $K = G/(\mathbb{Z}/k\mathbb{Z})$. Specifically, when K is the tetrahedral group we get 4 primitive complex reflection groups, when K is the octahedral group we get 8 and finally for the icosahedral group we get 7. Giving our 19 primitive complex reflection groups of rank 2.

4.3.2 Rank ≥ 3

The primitive complex reflection groups of rank ≥ 3 give rise to 15 exceptional cases and the symmetric groups (one of three infinite families). These are classified in a similar way to the real case, but instead of using Coxeter graphs, associated root graphs (first described by Cohen in [15] as an extension of Coxeter graphs) are used.

Chapter 5

Computational Results for Complex Reflection Groups

> Problems worthy of attack prove their worth by hitting back.
>
> ---
>
> Piet Hein [19, pg. 401]

5.1 GAP

GAP (Groups, Algorithms, Programming) [38] is a program for computational discrete algebra, which is particularly useful for computational group theory. It is a text based computer algebra system which includes a large number of functions that implement various algebraic algorithms. A vast number of these are with regards to groups, and, with the addition of an additional package, you can access algorithms regarding complex reflection groups. This package is called CHEVIE and was developed by Geck, Hiß, Lübeck, Malle and Pfeiffer [21]. Crucially for us the CHEVIE package contains the command `ComplexReflectionGroup` which gives each of the complex reflection groups. The input takes one of two forms, either an integer in the range $4, 5, \ldots, 37$ corresponding to one of the 34 exceptional groups, or it takes a triple (m, p, n) corresponding to a specific group in one of the infinite families. Furthermore CHEVIE includes `Reflections` as a command, which, as an output, gives a set of reflections from which we can find all reflections from a particular complex reflection group.

5.2 Determining the Reflections of a Complex Reflection Group

Most computations we want to do with complex reflection groups will first require us to actually have all the reflections of a particular group rather than just the elements. So to begin this chapter we present a program for creating a set of the reflections for any complex reflection group as permutations.

We first define a function, `ref`, which we will use to form the set of reflections.

```
gap> ref:=function(W,a)
   > local i,Ref;
   > Ref:=Reflections(W);
   > for i in Ref do
   > Add(a,i);
   > Add(a,i^2);
   > Add(a,i^3);
   > Add(a,i^4);
   > od;
   > return a;
```

```
> end;
```

To use this function we first need to specify the complex reflection group we want to find the reflections of. Define an empty list, run the program and then turn the outputted list into a set (since we will add some reflections more than once), this set will contain the reflections of the group. However it will also contain the identity, which as in Definition 4.1.1 we do not count as a reflection, so we must remove this.

The justification for this code is that the function `Reflections(W)` outputs a number of reflections, but not the powers of these reflections. We know that the highest order of a reflection in an exceptional complex reflection group is 5 (via the classification) and that is why in the function we only add up to a 4th power. Clearly this will add a number of reflections more than once, and if the reflection group contains a reflection of an order less than 5 then this function will add the identity. Thus, after calling the function, we turn the list into a set and remove the identity using:

```
gap> a:=Set(a);;
gap> RemoveSet(a,());;
```

Example 5.2.1. For an example we consider the complex reflection group G_5 (that is the group in Table 4.1 with Shephard-Todd number 5). First we let `W` be such a group, and `a` be an empty list:

```
gap> W:=ComplexReflectionGroup(5);;
gap> a:=[];;
```

Then apply the function to add the reflections to the list `a`.

```
gap> ref(W,a);;
```

Finally, our list `a` contains the identity, and a number of the reflections are repeated. So to get the complete set of reflections we turn `a` into a set and removing the identity.

```
gap> a:=Set(a);;RemoveSet(a,());;
```

The set `a` then contains all the reflections of G_5 as required, and Appendix A is that output.

5.3 Finding the Order of Reflections

As we can see in Table 4.1, the number and order of reflections in each group are known. Note that $2^6, 3^{16}, 4^{12}$, as in G_{10}, means that there are 6 reflections of order 2, 16 reflections of order 3 and 12 reflections of order 4. For G_2 and G_3 by $\phi(d)$ we mean the Euler-phi function which is the number of positive integers less than or equal to d that are coprime to d. In Section 5.2 we created a program to form a set of all reflections for that group. Here we create a program to calculate the order of each reflection in this set. We give such a program, named `reforder`, in Appendix B. The input for this program is the set `a` of all reflections for the particular group which we get from our previous function `ref`.

This function organises the reflections in the set `a` into subsets, each one containing reflections of the same order. Then for each possible value the program outputs if the group contains a reflection of that order, and how many such reflections there are.

Example 5.3.1. For a quick example we show how to find the order of the reflections in the group G_{11}.

First, in the same way as in Example 5.2.1, we create a set `a` with all the reflections in. We do this by defining an empty list, `a`, the specific complex reflection group `W` that want and then apply the `ref` function.

```
gap> W:=ComplexReflectionGroup(11);;
   > a:=[];;
   > ref(W,a);;
   > a:=Set(a);;
   > RemoveSet(a,());;
```

This `a` now contains all the reflections of G_{11}. We find the order of all the reflections by using the function `reforder` (which must have been previously defined by the user). Below we give the output that GAP would give:

```
gap> reforder(a);

The reflections in G11 are
2^18, 3^16, 4^12,
```

Which is exactly what we would expect compared to Table 4.1.

5.4 Well-Generated Complex Reflection Groups

Once again we refer to Table 4.1, in particular to the ‘rank’ column. Recall that the rank of a complex reflection group is the dimension of the complex vector space on which the group acts. The following theorem tells us that we need at most one more than the rank number of reflections in order to generate any irreducible complex reflection group. The Cartan-Dieudonné theorem says exactly this for real reflection groups, namely that they can be generated with n reflections.

Theorem 5.4.1. *If the rank of an irreducible complex reflection group, G, is n then the group G is generated by n or $n+1$ reflections.*

Proof. For the infinite families consider a group $G(m,p,n)$ and let $\{e_1,\ldots,e_n\}$ be a basis of the corresponding vector space V. We split the proof into two cases.

Case 1: $p=1$ or $p=m$

In this case we take the following $n-1$ reflections of order 2 with roots $e_1-e_2, e_2-e_3, \ldots, e_{n-1}-e_n$ and if $p=1$ a reflection of order m with root e_1 or if $p=m$ a reflection of order 2 with root $e_1 - e^{\frac{2\pi i}{m}} e_2$.

Case 2: $p \neq 1$ and $p \neq m$

Take the n generating reflections for $G(m,m,n)$ (as we did in Case 1) and then to generate the whole of $G(m,p,n)$ add an extra reflection of order $\frac{m}{p}$ with root e_1. Resulting in a generating set of reflections of size $n+1$.

For the 34 exceptional complex reflection groups we get that this result holds via the classification (as given in [15]). □

Motivated by this theorem we get the definition of a well-generated irreducible complex reflection group.

Definition 5.4.2. An irreducible complex reflection group G, of rank n, is *well-generated* if it is generated by n reflections.
A reducible complex reflection group is said to be *well-generated* if it is the product of well-generated irreducible complex reflection groups.

Again, it is well known which of the complex reflection groups are well-generated, and for a more elegant argument than we present here, regarding the degrees of a complex reflection group, we refer the reader to [6]. Here our proof again uses GAP.

We define the function `wellgen` given in Appendix C. This function, as its input, uses the particular complex reflection group W and the set of reflections, as found in Section 5.2 using `ref`. The output of this function is simply text saying whether or not the group is well-generated. Note that the program uses a standard GAP function `Combinations`, which is basically a 'power set' function, that is, it forms all subsets of particular size of any given set. In this program we find all subsets of reflections with size equal to the rank of the reflection group. The function then checks whether the group can be generated by one of these sets.

Example 5.4.3. We use this function in a similar way to previously, first defining the complex reflection group, an empty list, and using the `ref` function.

```
gap> W:=ComplexReflectionGroup(11);;
   > a:=[];;
   > ref(W,a);;
   > a:=Set(a);;
   > RemoveSet(a,());;
```

We now use our new function `wellgen` and give the output:

```
gap> wellgen(W);

G11 is not well-generated.
```

We can apply this function to any of the exceptional complex reflection groups, and we get the Table 5.1 of which of these are well-generated.

Table 5.1: Well-Generated Complex reflection groups

S-T Number	Well-Generated?	S-T Number	Well-Generated?
1	Yes	20	Yes
2	Yes if $p = 1$ or m	21	Yes
3	Yes	22	No
4	Yes	23	Yes
5	Yes	24	Yes
6	Yes	25	Yes
7	No	26	Yes
8	Yes	27	Yes
9	Yes	28	Yes
10	Yes	29	Yes
11	No	30	Yes
12	No	31	No
13	No	32	Yes
14	Yes	33	Yes
15	No	34	Yes
16	Yes	35	Yes
17	Yes	36	Yes
18	Yes	37	Yes
19	No		

5.5 Order of Reflections Needed to Generate a Complex Reflection Group

In this section we consider which reflections from a complex reflection group are actually needed to generate the group. We first consider generating sets of minimum size (i.e. either with the rank or rank plus one number of elements). From the previous section we know which are well-generated and so we know how many reflections we will need to generate each group. Clearly we need not consider the groups G_m for $m \in \{1, 4, 5, 12, 13, 16, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37\}$ as they only contain reflections of one order. It is then useful to know which of the remaining groups are well-generated and which are not, we can get this information from our previous work (the well-generated groups have Shephard-Todd numbers $6, 8, 9, 10, 14, 17, 18, 21$ and 26).

The program we use is called `requiredorders` and is considerably longer than our other programs. However, it is no more complicated, there are simply a lot more possibilities and so there is a lot of repetition in the code. It can be found in Appendix D. The program once again is called in a similar way, however, so the code can be used more universally, we add an extra argument, that is the number of reflections required to generate the given reflection group. As previously mentioned this will either be the rank of the group or the rank plus one. As well as the number of reflections needed to generated the group, like our other functions, `requiredorders` takes as its input the particular group, W, and the set of all reflections of the group. We give two examples to demonstrate this, one for a well-generated group, say G_{10}, and a not well-generated group, G_{11}.

Example 5.5.1. We first give an example for a well-generated group, G_{10}, and so we define a variable, `l`, to be equal to the rank of the group, which is the number of reflections needed to generated the group.

```
gap> W:=ComplexReflectionGroup(10);;
   > a:=[];;
   > ref(W,a);;
   > a:=Set(a);;
   > RemoveSet(a,());;
   > l:=Rank(W);;
   > requiredorders(W,a,l);

G10 cannot be generated solely by reflections of order 2
G10 cannot be generated solely by reflections of order 3
G10 cannot be generated solely by reflections of order 4
G10 cannot be generated by reflections of orders 2 and 3
G10 cannot be generated by reflections of orders 2 and 4
G10 can be generated by reflections of orders 3 and 4
```

So we see that G_{10} can be generated by a reflection of order 3 and a reflection of order 4. You do not need a reflection of order 2.

Example 5.5.2. We now do the same as above, but for G_{11}, which is not well-generated. So this time we set `l` to equal the rank plus one.

```
gap> W:=ComplexReflectionGroup(11);;
   > a:=[];;
   > ref(W,a);;
   > a:=Set(a);;
   > RemoveSet(a,());;
   > l:=Rank(W)+1;;
   > requiredorders(W,a,l);

G11 cannot be generated solely by reflections of order 2
```

```
G11 cannot be generated solely by reflections of order 3
G11 cannot be generated solely by reflections of order 4
G11 cannot be generated by reflections of orders 2 and 3
G11 cannot be generated by reflections of orders 2 and 4
G11 cannot be generated by reflections of orders 3 and 4
```

Here we see that to generate G_{11} you need one reflection of each order.

Using the above method we can work out which reflections are needed for which complex reflection groups. Indeed, in all but two cases you need at least one reflection of each order! The two exceptional cases are G_8 (generated by two reflections of order 4) and G_{10} (generated by a reflection of order 3 and a reflection of order 4). G_8 is particularly interesting when it is compared to G_9, they are both well-generated and both contain reflections of orders 2 and 4. However G_8 can be generated by reflections of order 4, whereas G_9 can not, it needs reflections of orders 2 and 4.

Chapter 6

Conclusion

> I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.
>
> Sir Isaac Newton [23, pg. 231]

6.1 Recent Developments

As mentioned in the abstract, the theory of complex reflection groups is an active area of research. The classification argument used in Chapter 4 is due to Cohen, from 1976, and since then there have been a number of developments. Most of these discussed here have been to try and create a universal theory of roots and root systems.

6.1.1 Arjeh Cohen

Although the complex reflection groups were first classified by Shephard and Todd in 1954 the classification argument that is mostly used today, including the one in this project, is that of Cohen. In his 1976 paper he presented root systems for complex reflection groups of dimension greater than or equal to 3. These systems were quite different from the systems in the real case, and they do not work in the same way. For example, there is no length function in this system.

6.1.2 Mervyn Hughes

Hughes has published a series of papers in the area of complex reflection groups [27], including root graphs [28] and his PhD thesis in the representations of such groups [26].

In 2001, Hughes with Alun Morris [29] gave a theory of root systems for well-generated two dimensional reflection groups, expanding Cohen's work in this area. This was done by exploring root graphs, as introduced by Coxeter in 1967 [18], and further studied by Hughes in 1999 [28], by developing a new way of classifying these finite reflection groups via particular polynomials with all their roots in $(0, 1)$. The irreducible well-generated two dimensional complex reflection groups have a one-to-one correspondence with such polynomials. Although this is a working system, it is not completely analogues to root systems for real reflection groups, and it still does not give rise to a corresponding idea of the length of an element.

6.1.3 Himmet Can

Can is another person who has published a large number of papers regarding complex reflection groups, and similarly to Hughes, on a variety of topics. Representations of $G(m,1,n)$ [10], combinatorial results [11] and, with Hawkins, a method for obtaining subsystems via a computer [13].

In [27] and [28] Hughes developed what he called 'extended Cohen diagrams', building upon Cohen's work in [15] to get subsystems of complex root systems. However Hughes' method did not work in every case, and this was what Can hoped to improve upon in his 2006 paper [12], where he classified proper subsystems.

6.1.4 Kirsten Bremke and Gunter Malle

Bremke and Malle have published two papers together on the concept of a length function for complex reflection groups. Such a function, used in conjunction with a theory of root systems, could potentially be very valuable given its importance for real reflection groups.

Firstly in [4] they give a root system and length function for the reflection groups $G(e,1,n)$ and in [5] this is extended to $G(e,e,n)$. When e is even this length function is completely analogous to the case for real reflection groups (this appears as Proposition 1.19 in [5]). Clearly this is not a function for all possible complex reflection groups, but for the $G(e,e,n)$ it is a very powerful tool.

6.1.5 JianYi Shi and Li Wang

'Reflection subgroups and sub-root systems of the imprimitive complex reflection groups' by Wang and Shi [40] is the most recent paper that we consider here. In this paper they classify all irreducible reflection subgroups of the imprimitive complex reflection groups (the $G(m, p, n)$). They classify such subgroups in a way much more similar to what we have done in Chapter 3, forming graphs from reflection sets (this was introduced by Howlett and Shi in [25]). Not enough time has passed since the publication of the paper to comment on the usefulness of it, however, the use of associated graphs for complex reflection groups seems like a step in the right direction.

6.1.6 Gustav Lehrer and Donald Taylor

We conclude this section on modern developments with a book, rather than a series of papers. That is the 2009 work by Lehrer and Taylor, 'Unitary Reflection Groups' [34]. The book contains a series of results taking the reader from the earliest work on complex reflection groups (the classification as done by Shephard and Todd) up to very recent developments in the field (including root systems for the $G(m, p, n)$). They also give a huge number of exercises for readers of varying levels. I recommend the book to anyone looking to study complex reflection groups further.

The book also contains applications of complex reflection groups and this can be found in Appendix C of [34].

6.2 Completion of the Project

The aim of this project was to build up the theory of reflection groups, which we have done through the classification of both real and complex reflection groups.

A number of results have been explored regarding real reflection groups. In Chapter 2 we worked towards the presentation of all such groups (Theorem 2.3.1), i.e. a real reflection group W has presentation

$$\langle s_1, s_2, \ldots, s_n \,|\, (s_i s_j)^{m_{ij}} = 1 \rangle.$$

This was the culmination of a number of theorems and lemmas, starting with the development of the theory of root, positive and simple systems.

The classification of real reflection groups (or finite Coxeter groups) was presented in Chapter 3 via their Coxeter graphs, namely the ones from Figure 3.12. In Chapter 4 the concept of a reflection was extended to that of a complex reflection. Here we discussed that there are in fact three infinite families of complex reflection groups, and 34 exceptional cases. The proof of the classification of the imprimitive reflection groups was given, and this was done in the same manner as Cohen in 1976 ([15]). Although the complete proof of the primitive reflection groups was not given, the outline of the idea was discussed and the complete proof referenced.

Chapter 5 contained a number of computational results which had been calculated via GAP with regards to complex reflection groups. Ranging from which are well-generated to what order reflections are required to generate them. The code for the various used programs appear in the appendices.

The project has been concluded by discussing recent developments in the field, referring to specific authors and papers. For further study, as a continuation of this project, one could extend the ideas of Section 5.5. In this section it was investigated what order of reflections, in a minimum generating set, are needed to generate the whole group. So, for example, for the complex reflection group G_{11}, which is not well-generated, we saw from Example 5.5.2 that the group required one reflection of each order (2, 3 and 4) to generate the group. However, we could drop the requirement that we consider a minimal generating set, and then check again if the group needs a reflection of each order. As an example, can G_{11} be generated by three reflections of order 2 and two reflections of order 3 say? The program given in Appendix D could easily, and quickly, be modified in order to calculate this. However, instead of checking at most the rank plus one number of reflections we would check up to the total number of reflections. The obvious problem here would be the added computational time. Thus, it would seem that to perform such a check computationally one would require a new program or access to more computation time.

Appendix A

GAP Result: Reflections of G_5

```
gap> [ (2,26,30)(4, 6,41)(5,27,32)(7,44,11)(8,34,39)
(10,12,16)(13,19,25)(14,22,46)(15,43,45)(17,36,29)(18,37,48)
(20,40,47)(24,42,28)(31,35,38),

   > (2,30,26)(4,41,6)(5,32,27)(7,11,44)(8,39,34)(10,16,12)
(13,25,19)(14,46,22)(15,45,43)(17,29,36)(18,48,37)(20,47,40)
(24,28,42)(31,38,35),

   > (1,3,9)(2,4,10)(5,11,25)(6,12,26)(7,13,27)(8,14,28)
(15,29,38)(16,30,41)(17,31,43)(19,32,44)(21,23,33)(22,24,34)
(35,45,36)(39,46,42),

   > (1,5,15)(2,6,16)(3,7,17)(4,8,18)(9,19,35)(10,20,34)
(11,21,36)(12,22,37)(13,23,38)(14,24,39)(26,40,46)(28,41,47)
(30,42,48)(32,33,43),

   > (1,9,3)(2,10,4)(5,25,11)(6,26,12)(7,27,13)(8,28,14)
(15,38,29)(16,41,30)(17,43,31)(19,44,32)(21,33,23)(22,34,24)
(35,36,45)(39,42,46),

   > (1,13,29)(2,47,42)(3,32,31)(5,19,7)(6,20,8)(9,11,45)
(10,39,37)(14,48,26)(15,25,21)(16,40,22)(17,27,23)(18,41,24)
(33,35,44)(36,43,38),

   > (1,15,5)(2,16,6)(3,17,7)(4,18,8)(9,35,19)(10,34,20)
```

```
(11,36,21)(12,37,22)(13,38,23)(14,39,24)(26,46,40)(28,47,41)
(30,48,42)(32,43,33),

  > (1,17,19)(3,35,5)(4,37,42)(7,9,15)(8,12,48)(10,46,47)
(11,38,33)(13,43,21)(18,22,30)(20,26,28)(23,32,36)(25,27,44)
(29,31,45)(34,40,41),

  > (1,19,17)(3,5,35)(4,42,37)(7,15,9)(8,48,12)(10,47,46)
(11,33,38)(13,21,43)(18,30,22)(20,28,26)(23,36,32)(25,44,27)
(29,45,31)(34,41,40),

  > (1,25,36)(2,46,37)(3,27,38)(4,47,39)(5,23,29)(6,28,48)
(7,33,31)(9,44,43)(11,32,13)(12,20,14)(15,35,17)(16,34,18)
(19,21,45)(24,30,40),

  > (1,27,43)(2,28,18)(3,44,36)(4,20,24)(5,33,45)(6,34,37)
(7,21,29)(8,22,42)(9,25,38)(10,26,41)(12,40,39)(14,30,47)
(16,46,48)(19,23,31),

  > (1,29,13)(2,42,47)(3,31,32)(5,7,19)(6,8,20)(9,45,11)
(10,37,39)(14,26,48)(15,21,25)(16,22,40)(17,23,27)(18,24,41)
(33,44,35)(36,38,43),

  > (1,32,45)(2,20,22)(3,11,29)(4,12,30)(6,40,42)(8,16,47)
(9,13,31)(10,14,18)(15,27,33)(17,44,21)(23,35,25)(24,37,26)
(28,34,46)(39,48,41),

  > (1,36,25)(2,37,46)(3,38,27)(4,39,47)(5,29,23)(6,48,28)
(7,31,33)(9,43,44)(11,13,32)(12,14,20)(15,17,35)(16,18,34)
(19,45,21)(24,40,30),

  > (1,43,27)(2,18,28)(3,36,44)(4,24,20)(5,45,33)(6,37,34)
(7,29,21)(8,42,22)(9,38,25)(10,41,26)(12,39,40)(14,47,30)
(16,48,46)(19,31,23),

  > (1,45,32)(2,22,20)(3,29,11)(4,30,12)(6,42,40)(8,47,16)
(9,31,13)(10,18,14)(15,33,27)(17,21,44)(23,25,35)(24,26,37)
(28,46,34)(39,41,48) ]
```

Appendix B

GAP Code: `reforder`

```
gap> reforder:=function(a)
   > local m,n,o,p,i;
   > m:=[];
   > n:=[];
   > o:=[];
   > p:=[];
   > for i in a do
   > if i^2=() then Add(m,i);fi;
   > if i^3=() then Add(n,i);fi;
   > if i^4=() and i^2<>() then Add(o,i);fi;
   > if i^5=() then Add(p,i);fi;
   > od;
   > Print("\n","The reflections in ",ReflectionName," are ");
   > if Length(m)>0 then Print("2^",Length(m),", ");fi;
   > if Length(n)>0 then Print("3^",Length(n),", ");fi;
   > if Length(o)>0 then Print("4^",Length(o),", ");fi;
   > if Length(p)>0 then Print("5^",Length(p),", ");fi;
   > Print("\n");
   > end;
```

Appendix C

GAP Code: `wellgen`

```
gap> wellgen:=function(W,a)
   > local l,b,i,c;
   > l:=Rank(W);
   > b:=Combinations(a,l);
   > c:=[];
   > for i in [1..Length(b)] do
   > if W=Subgroup(W,b[i]) then Add(c,i);fi;
   > od;
   > if Size(c)<>0 then Print("\n",ReflectionName(W)," is
     well-generated.","\n");
   > else Print("\n",ReflectionName(W)," is not well-generated.
     ","\n");
   > fi;
   > end;
```

Appendix D

GAP Code: requiredorders

```
gap> requiredorders:=function(W,a,l)
   > local m,n,o,p,i,b,c,d,e,f;
   > m:=[];
   > n:=[];
   > o:=[];
   > p:=[];
   > for i in a do
   > if i^2=() then Add(m,i);fi;
   > if i^3=() then Add(n,i);fi;
   > if i^4=() and i^2<>() then Add(o,i);fi;
   > if i^5=() then Add(p,i);fi;
   > od;
   >
   > if Length(m)>0 then
   >
   > b:=Combinations(m,l);
   > c:=[];
   > for i in [1..Length(b)] do
   > if W=Subgroup(W,b[i]) then Add(c,i);fi;
   > od;
   > if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated solely by reflections of order 2","\n");
```

```
   > else Print("\n",ReflectionName(W)," cannot be generated
solely by reflections of order 2","\n");
   > fi;
   >
   > fi;
   >
   > if Length(n)>0 then
   >
   > b:=Combinations(n,l);
   > c:=[];
   > for i in [1..Length(b)] do
   > if W=Subgroup(W,b[i]) then Add(c,i);fi;
   > od;
   > if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated solely by reflections of order 3","\n");
   > else Print("\n",ReflectionName(W)," cannot be generated
solely by reflections of order 3","\n");
   > fi;
   >
   > fi;
   >
   > if Length(o)>0 then
   >
   > b:=Combinations(o,l);
   > c:=[];
   > for i in [1..Length(b)] do
   > if W=Subgroup(W,b[i]) then Add(c,i);fi;
   > od;
   > if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated solely by reflections of order 4","\n");
   > else Print("\n",ReflectionName(W)," cannot be generated
solely by reflections of order 4","\n");
   > fi;
   >
   > fi;
   >
   > if Length(p)>0 then
   >
```

```
> b:=Combinations(p,l);
> c:=[];
> for i in [1..Length(b)] do
> if W=Subgroup(W,b[i]) then Add(c,i);fi;
> od;
> if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated solely by reflections of order 5","\n");
> else Print("\n",ReflectionName(W)," cannot be generated
solely by reflections of order 5","\n");
> fi;
>
> fi;
>
> if Length(m)>0 and Length(n)>0 and Length(o)>0 then
>
> d:=[];
> for i in [1..Length(m)] do
> Add(d,m[i]);
> od;
> for i in [1..Length(n)] do
> Add(d,n[i]);
> od;
> b:=Combinations(d,l);
> c:=[];
> for i in [1..Length(b)] do
> if W=Subgroup(W,b[i]) then Add(c,i);fi;
> od;
> if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated by reflections of orders 2 and 3","\n");
> else Print("\n",ReflectionName(W)," cannot be generated by
reflections of orders 2 and 3","\n");
> fi;
>
> e:=[];
> for i in [1..Length(m)] do
> Add(e,m[i]);
> od;
> for i in [1..Length(o)] do
```

```
> Add(e,o[i]);
> od;
> b:=Combinations(e,l);
> c:=[];
> for i in [1..Length(b)] do
> if W=Subgroup(W,b[i]) then Add(c,i);fi;
> od;
> if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated by reflections of orders 2 and 4","\n");
> else Print("\n",ReflectionName(W)," cannot be generated by
reflections of orders 2 and 4","\n");
> fi;
>
> f:=[];
> for i in [1..Length(n)] do
> Add(f,n[i]);
> od;
> for i in [1..Length(o)] do
> Add(f,o[i]);
> od;
> b:=Combinations(f,l);
> c:=[];
> for i in [1..Length(b)] do
> if W=Subgroup(W,b[i]) then Add(c,i);fi;
> od;
> if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated by reflections of orders 3 and 4","\n");
> else Print("\n",ReflectionName(W)," cannot be generated by
reflections of orders 3 and 4","\n");
> fi;
>
> fi;
>
> if Length(m)>0 and Length(n)>0 and Length(p)>0 then
>
> d:=[];
> for i in [1..Length(m)] do
> Add(d,m[i]);
```

```
> od;
> for i in [1..Length(n)] do
> Add(d,n[i]);
> od;
> b:=Combinations(d,l);
> c:=[];
> for i in [1..Length(b)] do
> if W=Subgroup(W,b[i]) then Add(c,i);fi;
> od;
> if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated by reflections of orders 2 and 3","\n");
> else Print("\n",ReflectionName(W)," cannot be generated by
reflections of orders 2 and 3","\n");
> fi;
>
> e:=[];
> for i in [1..Length(m)] do
> Add(e,m[i]);
> od;
> for i in [1..Length(p)] do
> Add(e,p[i]);
> od;
> b:=Combinations(e,l);
> c:=[];
> for i in [1..Length(b)] do
> if W=Subgroup(W,b[i]) then Add(c,i);fi;
> od;
> if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated by reflections of orders 2 and 5","\n");
> else Print("\n",ReflectionName(W)," cannot be generated by
reflections of orders 2 and 5","\n");
> fi;
>
> f:=[];
> for i in [1..Length(n)] do
> Add(f,n[i]);
> od;
> for i in [1..Length(p)] do
```

```
> Add(f,p[i]);
> od;
> b:=Combinations(f,l);
> c:=[];
> for i in [1..Length(b)] do
> if W=Subgroup(W,b[i]) then Add(c,i);fi;
> od;
> if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated by reflections of orders 3 and 5","\n");
> else Print("\n",ReflectionName(W)," cannot be generated by
reflections of orders 3 and 5","\n");
> fi;
>
> fi;
>
> if Length(n)>0 and Length(o)>0 and Length(p)>0 then
>
> d:=[];
> for i in [1..Length(n)] do
> Add(d,n[i]);
> od;
> for i in [1..Length(o)] do
> Add(d,o[i]);
> od;
> b:=Combinations(d,l);
> c:=[];
> for i in [1..Length(b)] do
> if W=Subgroup(W,b[i]) then Add(c,i);fi;
> od;
> if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated by reflections of orders 3 and 4","\n");
> else Print("\n",ReflectionName(W)," cannot be generated by
reflections of orders 3 and 4","\n");
> fi;
>
> e:=[];
> for i in [1..Length(n)] do
> Add(e,n[i]);
```

```
> od;
> for i in [1..Length(p)] do
> Add(e,p[i]);
> od;
> b:=Combinations(e,l);
> c:=[];
> for i in [1..Length(b)] do
> if W=Subgroup(W,b[i]) then Add(c,i);fi;
> od;
> if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated by reflections of orders 3 and 5","\n");
> else Print("\n",ReflectionName(W)," cannot be generated by
reflections of orders 3 and 5","\n");
> fi;
>
> f:=[];
> for i in [1..Length(o)] do
> Add(f,o[i]);
> od;
> for i in [1..Length(p)] do
> Add(f,p[i]);
> od;
> b:=Combinations(f,l);
> c:=[];
> for i in [1..Length(b)] do
> if W=Subgroup(W,b[i]) then Add(c,i);fi;
> od;
> if Size(c)<>0 then Print("\n",ReflectionName(W)," can be
generated by reflections of orders 4 and 5","\n");
> else Print("\n",ReflectionName(W)," cannot be generated by
reflections of orders 4 and 5","\n");
> fi;
>
> fi;
> end;
```

Bibliography

[1] C. T. Benson and L. C. Grove. *Finite Reflection Groups*. Springer, New York City, NY, 1971.

[2] M. Bhattacharjee, D. Macpherson, R. Möller, and P. Neumann. *Notes on Infinite Permutation Groups*. Springer-Verlag, Berlin, Germany, 1998.

[3] A. Borovik and A. V. Borovik. *Mirrors and Reflections: The Geometry of Finite Reflection Groups*. Springer, New York City, NY, 2009.

[4] K. Bremke and G. Malle. Reduced Words and a Length Function for $G(e, 1, n)$. *Indagationes Mathematicae*, 8:453–469, 1997.

[5] K. Bremke and G. Malle. Root Systems and Length Functions. *Geometriae Dedicata*, 72:83–97, 1998.

[6] M. Broué, G. Malle, and R. Rouquier. Complex Reflection Groups, Braid Groups, Hecke Algebras. *J. Reine Angew. Math.*, 500:127–190, 1998.

[7] F. Buekenhout and A. M. Cohen. *Diagram Geometry*. Springer-Verlag, New York City, NY, 2010.

[8] R. P. Burn. *Groups: A Path to Geometry*. Cambridge University Press, Cambridge, United Kindgom, 1985.

[9] S. Butler. *The Note-Books of Samuel Butler*. The Echo Library, Teddington, United Kingdom, 2006.

[10] H. Can. Representations of the Imprimitive Complex Reflection Groups $G(m, 1, n)$. *Communications in Algebra*, 26:2371–2393, 1998.

[11] H. Can. Some Combinatorial Results for Complex Reflection Groups. *European Journal of Combinatorics*, 19:901–909, 1998.

[12] H. Can. A New Approach to the Construction of Subsystems of Complex Root Systems. *Indagationes Mathematicae*, 17:13–29, 2006.

[13] H. Can and L. Hawkins. A Computer Program for Obtaining Subsystems. *Ars Combinatoria*, 58:257–269, 2001.

[14] C. Chevalley. Invariants of Finite Groups Generated by Reflections. *American Journal of Mathematics*, 77:778–782, 1955.

[15] A. M. Cohen. Finite Complex Reflection Groups. *Ann. scient. Ec. Norm. Sup.*, 9:379–436, 1976.

[16] H. S. M. Coxeter. Discrete Groups Generated by Reflections. *The Annals of Mathematics*, 35:588–621, 1934.

[17] H. S. M. Coxeter. The Complete Enumeration of Finite Groups of the Form $R_i^2 = (R_iR_j)^{k_{ij}}$. *Journal of the London Mathematical Society*, 10:21–25, 1935.

[18] H.S.M. Coxeter. Finite Groups Generated by Unitary Reflections. *Abhandlugen Aus Dem Mathematischen Seminar Der Universität Hamburg*, 31:125–135, 1967.

[19] B. A. Davey and G. A. Grätzer. *General Lattice Theory*. Birkhäuser, Boston, MA, 2003.

[20] G. de B. Robinson. *The Foundations of Geometry*. University of Toronto Press, Toronto, Canada, 1959.

[21] M. Geck, G. Hiß, F. Lübeck, G. Malle, and G. Pfeiffer. *CHEVIE – A System For Computing and Processing Generic Character Tables – version 4*, 2010.

[22] G. T. Gilbert. Positive Definite Matrices and Sylvester's Criterion. *The American Mathematical Monthly*, 98:44–46, 1991.

[23] D. Gjertsen. *The Newton Handbook*. Routledge & Kegan Paul Inc., New York City, NY, 1986.

[24] K. Hoffman and R. Kunze. *Linear Algebra: Second Edition.* Prentice-Hall, Englewood Cliffs, NJ, 1971.

[25] R. B. Howlett and J. Y. Shi. Congruence Classes of Presentations for the Complex Reflection Groups $G(m, 1, n)$ and $G(m, m, n)$. *Indagationes Mathematica*, 16:267–288, 2005.

[26] M. C. Hughes. The Representations of Complex Reflection Groups. *PhD Thesis, University of Wales*, 1981.

[27] M. C. Hughes. Complex Reflection Groups. *Communications in Algebra*, 18:3999–4029, 1990.

[28] M. C. Hughes. Extended Root Graphs for Complex Reflection Groups. *Communications in Algebra*, 27:199–148, 1999.

[29] M. C. Hughes and A. O. Morris. Root Systems for Two Dimensional Complex Reflection Groups. *Séminaire Lotharingien de Combinatoire*, 45:18–36, 2001.

[30] J. E. Humphreys. *Reflection Groups and Coxeter Groups.* Cambridge University Press, Cambridge, United Kindgom, 1990.

[31] R. Kane. *Reflection Groups and Invariant Theory.* Springer-Verlag, New York City, NY, 2001.

[32] R. Kaye and R. Wilson. *Linear Algebra.* Oxford University Press, New York City, NY, 1998.

[33] J. M. Legaré. *Orta-Undis and Other Poems.* William D. Ticknor & Company, Boston, MA, 1848.

[34] G. I. Lehrer and D. E. Taylor. *Unitary Reflection Groups.* Cambridge University Press, Cambridge, United Kingdom, 2009.

[35] J. Michel. `http://www.math.jussieu.fr/~jmichel/gap3/` accessed 1st February 2011. Updated Version of GAP 3.

[36] J. Roe. *Elementary Geometry.* Oxford University Press, Oxford, United Kindgom, 1993.

[37] B. Russell. *Mysticism and Logic: and Other Essays*. Longmans, Green & Co., Harlow, United Kingdom, 1919.

[38] M. Schönert et al. *GAP – Groups, Algorithms, and Programming – version 3 release 4.4*. Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Rheinisch Westfälische Technische Hochschule Aachen, German, 1997.

[39] G. C. Shephard and J. A. Todd. Finite Unitary Reflection Groups. *Canadian Journal of Mathematics*, 6:274–304, 1954.

[40] J. Y. Shi and L. Wang. Reflection Subgroups and Sub-Root Systems of the Imprimitive Complex Reflection Groups. *Science China Mathematics*, 53:1–8, 2010.

[41] G. Strang. *Linear Algebra and Its Applications: Fourth Edition*. Thomas Brooks/Cole, Belmont, CA, 2006.

Index