Complex Reflection Groups

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Introduction

The aim of this talk is to discuss the classification of complex reflection groups. I will not discuss real reflection groups here, but if you are interested you should read ‘Reflection Groups and Coexter Groups’ by Humphreys or the lesser well know, but very well written, ‘Reflection Groups and Invariant Theory’ by Kane.

Anyone who has studied finite reflection groups (also known as Coxeter groups) will know they are classified using root systems (like the semisimple Lie algebras). However there is no universally accepted root system for complex reflection groups. As such the classification is quite different. First we need to extend the idea of a reflection to a complex reflection.

Complex Reflections

Throughout we let V be a finite dimensional vector space over $\mathbb{C}$.

Definition

A **reflection** on V is a diagonalizable linear isomorphism $s : V \to V$ of finite order, not the identity, but fixes pointwise a hyperplane $H \subseteq V$.

Definition

A group $G \leq GL(V)$ is called a **reflection group** if G is generated by reflections.

Complex Reflections

A pseudo-reflection is similar to on the previous slide, but we do not restrict V to be over $\mathbb{C}$.

For the classification of complex reflection groups we only need to consider irreducible complex reflection groups. In all that follows we always let G be a complex reflection group.

Definition

A complex reflection group $G \leq GL(V)$ is called **reducible** if the vector space V is a reducible, G-module, i.e. it has notrivial G-submodules. If not G is said to be an **irreducible** complex reflection group.

Reflection Root

Whilst there is no universal theory of root systems, we do use the idea of a root.

Definition

A **(unitary) root of a reflection** is an eigenvector of length 1 corresponding to the unique nontrivial eigenvalue of the reflection. We denote a reflection $s \in G$ with root α as α_s.

Let Ω be the set of all roots from a reflection group:

$$\Omega = \{\alpha_s \,|\, \alpha \text{ a (unitary) root corresponding to the reflection } s \in G\}.$$

Classification

The complex reflection groups were first classified by Shepherd and Todd in 1954 and reclassified by Cohen in 1976.

There are 37 finite irreducible complex reflection groups. Three infinite families $\bigcap\left(Sym(n), \mathbb{Z}/m\mathbb{Z}, G(m,p,n)\right)$ and 34 exceptional groups $(G_4, \ldots, G_{37})$.

3 Cases

We define the rank of a complex reflection group to be the dimension of the complex vector space which the group acts on. The rank 1 complex reflection groups are just the cyclic groups, and so we consider rank ≥ 2. These we split into three cases:

1. imprimitive complex reflection groups $\left(G(m,p,n)\right)$
2. primitive complex reflection groups of rank 2 $\left(19 \text{ exceptional}\right)$
3. primitive complex reflection groups of rank ≥ 3 $\left(15 \text{ exceptional and } Sym(n)\right)$

In this talk we will just go through the proof of case 1 and a brief idea of how cases 2 and 3 work. Check Cohen for the complete proof.

Imprimitive Groups

Definition

A group $G \leq GL(V)$ is called **imprimitive** if there exists a decomposition of $V = \bigoplus_{i=1}^{k} V_i$ for $k > 2$ and the $(V_i)_{1 \leq i \leq k}$ are nontrivial proper linear subspaces of V and are permuted transitively by G. We call the $(V_i)_{1 \leq i \leq k}$ a **system of imprimitivity**.

Proposition

The following proposition is needed for the proof of the classification of the imprimitive complex reflection groups:

Proposition

Let G be an irreducible imprimitive complex reflection group in V, where $dim\, V = n$ and let $(V_i)_{1 \leq i \leq k}$ be a system of imprimitivity for G then:

1. $dim\, V_i = 1$ for $i = 1, \ldots, k$
2. Let $s \in G$ be an arbitrary reflection
    1. $sV_i = V_i$ for $1 \leq i \leq n$, or
    2. there exists $i \neq j$, $1 \leq i, j \leq n$ such that any root α_s of s is contained in $V_i \bigoplus V_j$ where $sV_i = V_j$ and $sV_k = V_k$ for all $k \neq i, j$ and $s^2 = 1$

Proof

Proof of Proposition

1. Fix i such that $dim\, V_i > 1$. Since G is irreducible $\exists j \neq i$ and a reflection s such that $sV_i = V_j$. Thus $dim(V_j \cap V_i) > 0$ contradicting $V_i \cap V_j = \{0\}$ (since V_i are a system of imprimitivity). So $dim\, V_i = 1$ for $i = 1, \ldots, k$.

2. Let $s \in G$ be a reflection with root α_s, first assume $\langle \alpha_s, V_i \rangle \neq 0 \bigcap \langle \cdot, \cdot \rangle$ an inner product over $\mathbb{C}]$ for $i = 1, 2$. Let ξ be a nontrivial eigenvalue such that $\langle \alpha_s, V_i \rangle \neq 0$ and $\alpha_s \notin V_i$.

   Since the group is imprimitive, $s_V 1 = V_2$. Now consider $0 \neq x_i \in V_i$ (again $i = 1, 2$) where $sx_1 = x_2$ then $\exists j \in \{1, \ldots, n\}$ such that $s^2 x_1 \in V_j$ and thus $s^2 x_1 \in (\mathbb{C}\alpha_s + \mathbb{C}_x) \cap V_j = (V_1 \oplus V_2) \cap V_j$ so $j = 1$ or $j = 2$.

Proof (continued)

Proof of Proposition (continued...)

2. Since $\alpha_s \in V_1$ we know $s^2 x_1 = x_1$ and $\xi^2 = 1 \Rightarrow \xi = -1$ and thus $s^2 = 1$. Then as α_s is a scalar multiple of $(x_1 - x_2)$ we get $\alpha_s \in V_1 \oplus V_2)$. Now consider $\langle \alpha_s, V_i \rangle = 0$ for $i \geq 3$ and so $\exists j \geq 3$ such that $sV_i = V_j$. Thus
$\alpha_s \in [(V_i \oplus V_j) \cap (V_1 \oplus V_2)] = \{0\}$ which is impossible. Thus $\langle \alpha_s, V_i \rangle = 0$ for $i \geq 3$ and so $sV_k = V_k$ for $k \geq 3$. □

Infinite Family

Definition

For any $m, p, n \geq 1$ such that $p|m$ define $\mathbf{G(m, p, n)}$ to be the group of $n \times n$ monomial matrices with non-zero entries a_i such that the a_i are mth roots of unity and $\prod_{i=1}^{n} a$ is an $^m/_p$th root of unity.

Or equivalently, $G(m, p, n) = A(m, p, n) \rtimes \pi_n$ where $A(m, p, n)$ are diagonal matrices whose elements are mth roots of unity and for $A \in A(m, p, n)$, $(det\ A)^{\frac{m}{p}} = 1$ and π_n are $n \times n$ permutation matrices.

Remark

Braué, Malle and Rouquier use $G(de, e, r)$.

Examples

Examples

$$
\begin{aligned}
G(1,1,n) &= W(A_{n-1}) \\
G(2,1,n) &= W(B_n) \\
G(2,2,n) &= W(D_{2n}) \\
G(m,p,1) &= G({}^{m}/_{p},1,1) = \mathbb{Z}/({}^{m}/_{p})\mathbb{Z}
\end{aligned}
$$

Irreducibility Lemma

Lemma

$G(m,p,n)$ is irreducible if and only if $(m,p,n) \neq (2,2,2)$ and $m > 1$.

Proof

"$\Rightarrow$" Assume $G = G(m,p,n)$ is irreducible and suppose G fixes (leaves invariant) a nontrivial subspace $W \subset V$. Since W is π_n-invariant $W = \mathbb{C}(e_1 + \cdots + e_n)$ up to switching between W and $W^{\perp}$. Since $A(m,p,n)$ stabilizes $\mathbb{C}(e_1 + \cdots + e_n)$ all diagonal elements of $A(m,p,n)$ must be equal, i.e. we must have $(m,p,n) = (1,1,n)$ or $(2,2,2)$.

"$\Leftarrow$" $(m,p,n) = (1,1,n)$ $(Sym(n))$ or $(2,2,2)$ (Klein four-group) are reducible. $\square$

Conjugate Theorem

Theorem

If $\dim V = n \geq 2$ *and let* $G \leq GL(V)$ *be an irreducible complex reflection group then* G *is conjugate to* $G(m,p,n)$ *for* $m,p \in \mathbb{N}$ *and* $p|m$.

Proof

Let G be as above, then we can find an orthonormal basis $\{\epsilon_1, \ldots, \epsilon_n\}$ of V such that $V_i = \mathbb{C}\epsilon_i$ for $1 \leq i \leq n$ form a system of imprimitivity for G. Without changing the conjugacy class of G we see this basis equals $\{e_1, \ldots, e_n\}$. By our previous proposition $\pi_n \leq G$.

The cyclic group generated by reflections which fix a hyperplane (without loss of generality He_1) of V pointwise, and let the order of this group be q. Then $A(q,l,n) \leq G$.

Conjugate Theorem (continued)

Proof (continued...)

Now we consider the group $A(q,l,n) \rtimes \pi_n$. The only reflections of G which are not in it are $s' \in G$ such that $s'e_i = \phi e_j$ with $i \neq j$ and $s'e_k = e_k$ for all $k \neq i,j$. Take $i = 1$ and $j = 2$ and let $s = s_2 \in G$ be a reflection such that $se_1 = e_2$. Then,

$$(ss')e_1 = \phi e_1 \quad \text{and} \quad (ss')e_2 = \phi^{-1} e_2.$$

So ϕ is a root of unity. Now for another reflection $t \in G$ such that $tV_1 = V_2$, let m be the maximum order of elements $st \in G$, thus $A(m,m,n) \leq G$.

Now let $p^m/_q$ so $p|m$ and $A(m,p,n) = \langle A(q,l,n), A(m,m,n)\rangle$. So $G(m,p,n) := A(m,p,n) \rtimes \leq G$. But all reflections of G are contained in $A(m,p,n) \rtimes \pi_n$ so $G(m,p,n) = G$. $\square$

Fact

Due to time constrains this results is just stated as a fact:

Fact

If $G = G(m,p,n)$ is irreducible then G has a unique system of imprimitivity as long as:

$$G(m,p,n) \notin \Big\{(2,1,2),(4,4,2),(3,3,3),(2,2,4)\Big\}$$

.

This fact along with the other theorems completes the classification of imprimitive complex reflection groups.

Primitive Complex Reflection Groups of Rank 2

Part 2 of the classification of complex reflection groups looks at the primitive groups of rank 2. We require a group G generated by reflections such that

$$G \leq GL_2(\mathbb{C}) = \mathbb{C} \times SL_2(\mathbb{C}).$$

Recall the conjugacy classes of finite subgroups of $GL_2(\mathbb{C})$ are $\mathbb{Z}/m\mathbb{Z}$, D_{2m}, $\langle x, y \mid x^2 = y^3 = (xy)^k = 1 \rangle$ for $k = 3, 4, 5$ (tetrahedral, octahedral and icosahedral). Then for $K, H \leq SL_2(\mathbb{C})$ such that

$$H \triangleleft K \text{ and } K/H = \mathbb{Z}/m\mathbb{Z}.$$

Primitive Complex Reflection Groups of Rank 2

We can get all subgroups of $GL_2(\mathbb{C})$ by defining a group G such that $G \leq {}^{\mathbb{Z}}\!/_{km\mathbb{Z}} \times K \leq GL_2(\mathbb{C})$ for $k \geq 2$ and where ${}^{\mathbb{Z}}\!/_{k\mathbb{Z}} \triangleleft G$ and $K = {}^{G}\!/_{(\mathbb{Z}/k\mathbb{Z})}$.

- when K is tetrahedral we get 4 primitive complex reflection groups
- when K is octahedral we get 8 primitive complex reflection groups
- when K is icosahedral we get 7 primitive complex reflection groups

This gives 19 exceptional complex reflection groups.

Primitive Complex Reflection Groups of Rank 3

The primitive complex reflection groups of rank 3 gives rise to 15 exceptional groups and $Sym(n)$. The classification requires the use of **root graphs** which is an extension of a Coxeter graph. These were first introduced by Cohen.

Complex Reflection Group Generation

Theorem

All complex reflection groups can be generated by the rank or rank $+1$ *number of reflections.*

Definition

A complex reflection group is **well generated** if it is generated by the rank number of reflections.

Example

- $Sym(n)$ is well generated
- $G(m, p, n)$ is well generated if and only if $p = 1$ or $p = m$
- $\mathbb{Z}/m\mathbb{Z}$ is well generated
- G_r for $r \in \{4, 5, 6, 8, 9, 10, 14, 16, 17, 18, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37\}$ is well generated